# Structures for the category of torsion theories

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# Torsion theories

A **torsion theory** is a triple $(\mathbb{C}, \mathcal{T}, \mathcal{F})$, where $\mathbb{C}$ is a pointed category and $(\mathcal{T}, \mathcal{F})$ is a pair of full replete subcategories of $\mathbb{C}$, such that:

(T1) every morphism in $\mathbb{C}$ from an object in $\mathcal{T}$ to an object in $\mathcal{F}$ factors through 0;

(T2) for every $X \in \mathbb{C}$ there is a sequence of morphisms

$$T^X \xrightarrow{\ell^X} X \xrightarrow{r^X} F^X$$

such that $T^X \in \mathcal{T}$, $F^X \in \mathcal{F}$, $\ell^X = \mathrm{Ker}(r^X)$ and $r^X = \mathrm{Coker}(\ell^X)$.

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**Example.**

$(\mathbf{Ab}, \{\text{torsion abelian groups}\}, \{\text{torsion-free abelian groups}\})$

$$T(G) \xhookrightarrow{i} G \xrightarrow{\pi} G/T(G)$$

# Products of torsion theories

Consider the 2-category $\mathbf{Tor}^{ch}$ of torsion theories and functors preserving torsion objects, torsion free objects and sending the chosen short exact sequences to short exact sequences.

**Theorem.**

*Consider a non-empty family $\mathbb{C} = (\mathbb{C}_i)_{i \in I}$ of pointed categories and a family $(\mathcal{T}_i, \mathcal{F}_i)_{i \in I}$ of pairs of categories. The triple*

$$(\Pi\mathbb{C}, \Pi\mathcal{T}, \Pi\mathcal{F}) = \left( \prod_{i \in I} \mathbb{C}_i, \prod_{i \in I} \mathcal{T}_i, \prod_{i \in I} \mathcal{F}_i \right)$$

*is a torsion theory if and only if each $(\mathbb{C}_i, \mathcal{T}_i, \mathcal{F}_i)$ is a torsion theory.*

# A 2-monad for torsion theories

$$M: \textbf{PointCat} \longrightarrow \textbf{PointCat}$$

$$\begin{array}{c} \mathbb{C} \\ G \Big( \overset{\alpha}{\Longrightarrow} \Big) H \\ \mathbb{D} \end{array} \quad \mapsto \quad \begin{array}{c} \mathbb{C} \times \mathbb{C} \\ G \times G \Big( \overset{\alpha \times \alpha}{\Longrightarrow} \Big) H \times H \\ \mathbb{D} \times \mathbb{D} \end{array}$$

is a (strict) 2-monad, with unit given by diagonal functors $\eta_{\mathbb{C}} : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ and multiplication $\mu_{\mathbb{C}} = \pi_{1,4} : (\mathbb{C} \times \mathbb{C}) \times (\mathbb{C} \times \mathbb{C}) \to \mathbb{C} \times \mathbb{C}$ sending $((X, Y), (Z, W))$ to $(X, W)$;

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$$\left( \mathbb{C} \overset{G}{\underset{H}{\rightrightarrows}} \mathbb{D},\ \alpha \colon G \Rightarrow H \right) \mapsto \left( \mathbb{C}\times\mathbb{C} \overset{G\times G}{\underset{H\times H}{\rightrightarrows}} \mathbb{D}\times\mathbb{D},\ \alpha\times\alpha \colon G\times G \Rightarrow H\times H \right)$$

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All $M$-pseudo-algebras are torsion theories:
given $(\mathbb{C}, Q, Q_\mu, Q_\eta)$ we have $(\mathbb{C}, \mathcal{T}, \mathcal{F}) \in \mathbf{Tor}^{ch}$ with $\mathcal{T}$ given by the essential image of $\mathbb{C} \times \mathbf{0} \subseteq \mathbb{C} \times \mathbb{C} \xrightarrow{Q} \mathbb{C}$ and $\mathcal{F}$ given by the essential image of $\mathbf{0} \times \mathbb{C} \subseteq \mathbb{C} \times \mathbb{C} \xrightarrow{Q} \mathbb{C}$.

# Rectangular torsion theories

**Definition.**

A **rectangular torsion theory** is a torsion theory $(\mathbb{C}, \mathcal{T}, \mathcal{F})$ such that the canonical functor $\Gamma \colon \mathbb{C} \to \mathcal{T} \times \mathcal{F}$ sending $X$ to the pair $(T^X, F^X)$ is an equivalence of categories.

**Theorem.**

*The 2-category* $\mathbf{Tor}_R^{ch}$ *of rectangular torsion theories is 2-equivalent (over* **PointCat***) to the 2-category of M-pseudo-algebras.*

Given $(\mathbb{C}, \mathcal{T}, \mathcal{F})$ the corresponding pseudo-algebra map is the composite

$$\mathbb{C} \times \mathbb{C} \xrightarrow{\Gamma \times \Gamma} (\mathcal{T} \times \mathcal{F}) \times (\mathcal{T} \times \mathcal{F}) \xrightarrow{\pi_{1,4}} \mathcal{T} \times \mathcal{F} \xrightarrow{\Gamma'} \mathbb{C};$$

$$(X, Y) \mapsto ((T^X, F^X), (T^Y, F^Y)) \mapsto (T^X, F^Y) \mapsto \Gamma'((T^X, F^Y))$$

# Internal rectangular bands

The algebras for the monad on **Set** given by $X \mapsto X \times X$ are precisely the *rectangular bands*, i.e. idempotent semigroups satisfying $xyz = xz$.

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**Definition.**

Let $\mathfrak{L}$ be a 2-category with squares. An **internal rectangular band in the 2-category** $\mathfrak{L}$ is a pseudo-algebra for the 2-monad $\mathbb{C} \mapsto \mathbb{C} \times \mathbb{C}$ over $\mathfrak{L}$.

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**Theorem.**

*Every rectangular torsion theory is a product of the form* $(\mathbb{C}, \mathbb{C}, \mathbf{0}_{\mathbb{C}}) \times (\mathbb{D}, \mathbf{0}_{\mathbb{D}}, \mathbb{D})$ *in* $\mathbf{Tor}^{ch}$.

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**Theorem.**

*The 2-category* $\mathbf{Tor}^{ch}_{R}$ *is biequivalent to* **PointCat** $\times$ **PointCat**.

# Torsion theory classes of epimorphisms

Let $\mathbb{C}$ be a pointed category with binary products. Consider any full subcategory of epimorphisms $\mathcal{E}$ of the arrow category $\mathbb{C}^2$ containing all isomorphisms and all morphisms to a zero object.

**Theorem.**

$(\mathcal{E}, \{\textit{isomorphisms}\}, \{\textit{morphisms to a zero object}\})$ *is a rectangular torsion theory if and only if every object of* $\mathcal{E}$ *is a product projection in* $\mathbb{C}$.

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**Theorem.**

*The class of split epimorphisms in a pointed regular category is a rectangular torsion theory class if and only if every split epimorphism is a product projection.*

# Pretorsion theories

A **pretorsion theory** is a triple $(\mathbb{C}, \mathcal{T}, \mathcal{F})$, where $\mathbb{C}$ is a category and $(\mathcal{T}, \mathcal{F})$ is a pair of full replete subcategories, such that for the ideal $\mathcal{N}$ of morphisms that factor through objects in the intersection $\mathcal{T} \cap \mathcal{F}$:

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such that $T^X \in \mathcal{T}$, $F^X \in \mathcal{F}$, $\ell^X$ is a $\mathcal{N}$-kernel of $r^X$ and $r^X$ is a $\mathcal{N}$-cokernel of $\ell^X$.

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**Example.**

$(\mathbf{Preord}, \{\text{equivalence relations}\}, \{\text{partial orders}\})$

# The category of short exact sequences in $\mathbb{C}$

Let $\mathbb{C}$ be a category with all kernels and cokernels and let $\mathcal{N}$ be a closed ideal of (null) morphisms in $\mathbb{C}$.

We consider the category $\mathsf{Ses}(\mathbb{C})$.

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$$

We define a pretorsion theory $(\mathrm{Ses}(\mathbb{C}), \mathcal{T}, \mathcal{F})$, where

- $\mathcal{T}$ consists of s.e.s of the form $\bullet \cong \bullet \to \bullet$
- $\mathcal{F}$ consists of s.e.s of the form $\bullet \to \bullet \cong \bullet$

# Short exact sequences in Ses $(\mathbb{C})$

## Proposition.

$(u, v, w) = \mathrm{Ker}((u', v', w'))$ *iff* $u = \mathrm{Ker}(u')$ *and* $v = \mathrm{Ker}(v')$.

$(u', v', w') = \mathrm{Coker}((u, v, w))$ *iff* $v' = \mathrm{Coker}(v)$ *and* $w' = \mathrm{Coker}(w)$.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ X' & \underset{f'}{\longrightarrow} & Y' & \underset{g'}{\longrightarrow} & Z' \\ {\scriptstyle u'}\downarrow & & {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle w'} \\ X'' & \underset{f''}{\longrightarrow} & Y'' & \underset{g''}{\longrightarrow} & Z'' \end{array}$$

# Short exact sequences in $\mathrm{Ses}(\mathbb{C})$

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The s.e.s. associated to $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathrm{Ses}(\mathbb{C})$ is

$$\begin{array}{ccccc} X & = & X & \longrightarrow & \mathrm{Coker}(\mathrm{id}_X) \\ \| & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \exists!} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \\ {\scriptstyle \exists!}\downarrow & & {\scriptstyle g}\downarrow & & \| \\ \mathrm{Ker}(\mathrm{id}_Z) & \longrightarrow & Z & = & Z \end{array}$$

# A pseudocomonad for pretorsion theories

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- functors preserving kernels and cokernels
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Let **Clldl** be the 2-category of

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**Proposition.**

$$\Omega: \qquad \mathbf{Clldl} \longrightarrow \mathbf{Clldl}$$

$$\begin{array}{c} \mathbb{C} \\ G \left( \overset{\alpha}{\Longrightarrow} \right) H \\ \mathbb{D} \end{array} \quad \mapsto \quad \begin{array}{c} \mathsf{Ses}(\mathbb{C}) \\ G \left( \overset{\alpha}{\Longrightarrow} \right) H \\ \mathsf{Ses}(\mathbb{D}) \end{array}$$

*is a pseudocomonad, with counit sending $X \to Y \to Z$ to $Y$ and comultiplication selecting the chosen s.e.s of the pretorsion theory on* $\mathsf{Ses}(\mathbb{C})$.

# A pseudocomonad for pretorsion theories

All normal $\Omega$-pseudo-coalgebras are pretorsion theories:

given $((\mathbb{C}, \mathcal{N}), \lambda\colon \mathbb{C} \to \mathsf{Ses}(\mathbb{C}), \lambda_\delta\colon \delta_\mathbb{C} \circ \lambda \overset{\cong}{\Rrightarrow} \Omega(\lambda) \circ \lambda)$ we have $(\mathbb{C}, \mathcal{T}, \mathcal{F}) \in \mathbf{PTor}^{ck}$ with $\mathcal{T}$ the collection of objects $X \in \mathbb{C}$ such that $\lambda(X)$ is of the form $\bullet \cong \bullet \to \bullet$, $\mathcal{F}$ defined dually and chosen s.e.s. given by the coalgebra map $\lambda$.

The isomorphisms given by $\lambda_\delta$ ensure that the s.e.s. associated to $X$ is of the form $T \to X \to F$.

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The isomorphisms given by $\lambda_\delta$ ensure that the s.e.s. associated to $X$ is of the form $T \to X \to F$.

**Remark.**

The fact that $\lambda\colon \mathbb{C} \to \mathrm{Ses}(\mathbb{C})$ preserves kernels and cokernels implies that $\Omega$-pseudo-coalgebras are pretorsion theories with additional properties.

# Bihereditary pretorsion theories

**Definition.**

A pretorsion theory $(\mathbb{C}, \mathcal{T}, \mathcal{F})$ is said **hereditary** if the functor $\mathbb{C} \xrightarrow{T} \mathcal{T} \hookrightarrow \mathbb{C}$ preserves kernels and **cohereditary** if it has the dual property. It is said **bihereditary** if it is both hereditary and cohereditary.

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- Rectangular torsion theories are bihereditary.

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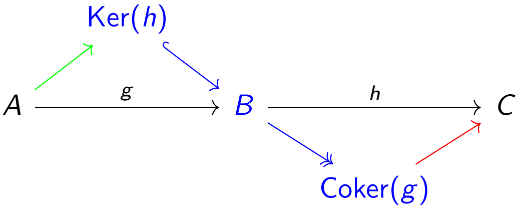
- Rectangular torsion theories are bihereditary.
- Given a finite preordered set $(X = \{1, 2, \dots, n\}, \leq)$ the triple $(X, T, F)$ is a pretorsion theory iff
  - (i) $T \cup F = X$;
  - (ii) $1 \in T$ and $n \in F$;
  - (ii) for every $i = 1, \dots, n-1$ if $i \in T$ and $i+1 \in F$, then either $i \in F$ or $i+1 \in T$.

  All such pretorsion theories are bihereditary.

# Exact sequences

## Definition.

We say that the sequence $A \xrightarrow{g} B \xrightarrow{h} C$ is **exact** if $h \circ g$ is null, the replecement sequence (in blue) is a short exact sequence, the morphism $A \to \mathrm{Ker}(h)$ coreflects null morphisms and the morphism $\mathrm{Coker}(g) \to C$ reflects null morphism.



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## Proposition.

$$\widetilde{\Omega}: \quad \mathbf{ClIdl}^{ex} \longrightarrow \mathbf{ClIdl}^{ex}$$

$$\begin{array}{c} \mathbb{C} \\ G \left( \overset{\alpha}{\Longrightarrow} \right) H \\ \mathbb{D} \end{array} \quad \mapsto \quad \begin{array}{c} \mathrm{Ses}(\mathbb{C}) \\ [G] \left( \overset{[\alpha]}{\Longrightarrow} \right) [H] \\ \mathrm{Ses}(\mathbb{D}) \end{array}$$

*where $[G]$ computes the short exact replacement of the image along $G$, is a pseudocomonad.*

# Generalized pretorsion theories

All pretorsion theories are normal $\widetilde{\Omega}$-pseudo-coalgebras!

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All pretorsion theories are normal $\widetilde{\Omega}$-pseudo-coalgebras!

But not all $\widetilde{\Omega}$-pseudo-coalgebras are pretorsion theories...

$$\lambda: \quad \mathbb{C} \longrightarrow \mathsf{Ses}(\mathbb{C})$$

$$\begin{array}{c} X \\ \downarrow h \\ Y \end{array} \quad \mapsto \quad \begin{array}{ccccc} T^X & \xrightarrow{\ell^X} & X & \xrightarrow{r^X} & F^X \\ \downarrow h^T & & \downarrow h & & \downarrow h^F \\ T^Y & \xrightarrow{\ell^Y} & Y & \xrightarrow{r^Y} & F^Y \end{array}$$

with $\boldsymbol{(\ell^X)^F}$ **null** and $\boldsymbol{(r^X)^T}$ **null** for every $X \in \mathbb{C}$
(with some additional coherence conditions).

# Can we characterize pretorsion theories among coalgebras?

Pretorsion theories are precisely the normal $\widetilde{\Omega}$-pseudo-coalgebras such that $\lambda\colon \mathbb{C} \to \mathsf{Ses}\,(\mathbb{C})$ **preserves the s.e.s. in its image**.

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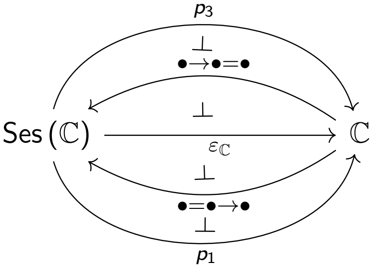
Pretorsion theories are precisely the normal $\widetilde{\Omega}$-pseudo-coalgebras such that $\lambda \colon \mathbb{C} \to \mathsf{Ses}(\mathbb{C})$ **preserves the s.e.s. in its image**.

Alternatively, we can characterize the pretorsion theories as the normal pseudo-coalgebras satisfying a certain equation in terms of the adjoints



# What's next?

- properties and examples of *generalized pretorsion theories*
- "simplicial" monads
- further understand limits and colimits of (pre)torsion theories
- other structures for categories of torsion theories
- internal rectangular bands in other categories

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