Closed Categories, Pro-operads and Goodwillie Calculus

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(joint work with Thomas Blom)

in progress

I. What is a pro-monoid ?

$(\mathcal{C}, \otimes, I)$: monoidal category

A monoid in $(\mathcal{C}, \otimes, I)$ consists of :

- object X in $\mathcal{C}$
- morphisms $X \otimes X \xrightarrow{\mu} X$, $I \xrightarrow{\eta} X$

s.t.

$$
\begin{tikzcd}
(X\otimes X)\otimes X \arrow[rr, "\cong"] \arrow[d, "\mu\otimes 1_X"'] & & X\otimes(X\otimes X) \arrow[d, "1_X\otimes\mu"] \\
X\otimes X \arrow[r, "\mu"'] & X & X\otimes X \arrow[l, "\mu"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes X \arrow[r, "\eta\otimes 1_X"] \arrow[dr, "\cong"'] & X\otimes X \arrow[d, "\mu"] & X\otimes I \arrow[l, "1_X\otimes\eta"'] \arrow[dl, "\cong"] \\
& X &
\end{tikzcd}
$$

commute.

A <u>pro-object</u> in category $\mathcal{C}$ is a diagram

$$X^\bullet : I \to \mathcal{C}$$

where I is a cofiltered category.

These form a category, $\mathrm{Pro}(\mathcal{C})$, with

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X^\bullet, Y^\bullet) := \lim_k \operatorname{colim}_j \mathrm{Hom}_{\mathcal{C}}(X^j, Y^k)$$

Defn. 1

A <u>pro-monoid</u> in $(\mathcal{C}, \otimes, I)$ is a pro-object in the category of monoids in $\mathcal{C}$: a pro-object $X^\bullet$ in $\mathcal{C}$ with

$$X^i \otimes X^i \to X^i$$

for each $i \in \mathbb{I}$, ...

Lemma

$\mathrm{Pro}(\mathcal{C})$ inherits a monoidal structure $\tilde{\otimes}$:

$X^\bullet : \mathbb{I} \to \mathcal{C}$, $Y^\bullet : \mathbb{J} \to \mathcal{C}$: pro-objects in $\mathcal{C}$

$X^\bullet \tilde{\otimes} Y^\bullet : \mathbb{I} \times \mathbb{J} \to \mathcal{C}$: $(i,j) \mapsto X^i \otimes Y^j$

Defn. 2

A <u>pro-monoid</u> in $\mathcal{C}$ is a monoid in $(\mathrm{Pro}(\mathcal{C}), \tilde{\otimes}, I^{\bullet})$

a pro-object $X^{\bullet}$ in $\mathcal{C}$ with pro-morphisms

$$X^{\bullet} \tilde{\otimes} X^{\bullet} \longrightarrow X^{\bullet} \qquad\qquad I^{\bullet} \longrightarrow X^{\bullet}$$

i.e. for each $k \in \mathbb{I}$, we have

$$X^{i} \otimes X^{j} \longrightarrow X^{k} \qquad \text{for some } i, j \in \mathbb{I}$$

In particular, each pro-monoid (Defn 1) is a pro-monoid (Defn. 2).

Suppose $(\mathcal{C}, \otimes, I)$ is <u>right-closed</u> monoidal with hom-objects

$$\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$$

Then $(\mathcal{C}, \mathrm{Map}_{\mathcal{C}}(-,-), I)$ is a <u>closed category</u> in the sense of Eilenberg-Kelly:

$$X \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{C}}(I, X) \qquad \mathrm{Map}_{\mathcal{C}}(X,Y) \xrightarrow{m_{XYZ}} \mathrm{Map}_{\mathcal{C}}(\mathrm{Map}_{\mathcal{C}}(Y,Z), \mathrm{Map}_{\mathcal{C}}(X,Z))$$

$$I \longrightarrow \mathrm{Map}_{\mathcal{C}}(X,X)$$

s.t.

$$\begin{tikzcd}
 & \mathrm{Map}_{\mathcal{C}}(X,X) \arrow[d, "m_{XXZ}"] \\
I \arrow[ur] \arrow[r] & \mathrm{Map}_{\mathcal{C}}(\mathrm{Map}_{\mathcal{C}}(X,Z), \mathrm{Map}_{\mathcal{C}}(X,Z))
\end{tikzcd}$$

, ... commute

Proposition [Blom-C]

Suppose $(\mathcal{C}, \mathrm{Map}_{\mathcal{C}}(-,-), I)$ is a closed category which admits filtered colimits.

Then $\mathrm{Pro}(\mathcal{C})$ inherits a closed structure :

for $X^\bullet : \mathbb{I} \to \mathcal{C}$, $Y^\bullet : \mathbb{J} \to \mathcal{C}$

define $\mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(X^\bullet, Y^\bullet) : \mathbb{J} \to \mathcal{C}$ by

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(X^\bullet, Y^\bullet)^j := \operatorname*{colim}_i \mathrm{Map}_{\mathcal{C}}(X^i, Y^j)$$

Def.

A **monoid** in a closed category $(\mathcal{C}, \mathrm{Map}_{\mathcal{C}}(-,-), I)$ consists of

- an object X in $\mathcal{C}$
- morphisms $X \xrightarrow{\tilde{\mu}} \mathrm{Map}_{\mathcal{C}}(X,X)$, $I \xrightarrow{\eta} X$

s.t.

$$
\begin{tikzcd}
X \arrow[r, "\tilde{\mu}"] \arrow[d, "\tilde{\mu}"'] & \mathrm{Map}_{\mathcal{C}}(X,X) \arrow[r, "m_{XXX}"] & \mathrm{Map}_{\mathcal{C}}(\mathrm{Map}_{\mathcal{C}}(X,X), \mathrm{Map}_{\mathcal{C}}(X,X)) \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(X,X) \arrow[rr] & & \mathrm{Map}_{\mathcal{C}}(X, \mathrm{Map}_{\mathcal{C}}(X,X))
\end{tikzcd}
$$

. . . . . commute.

Defn. 3

A <u>pro-monoid</u> in $\mathcal{C}$ is a monoid in the closed structure on $\mathrm{Pro}(\mathcal{C})$: a pro-object $X^\bullet$ with

$$X^\bullet \longrightarrow \mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(X^\bullet, X^\bullet) \quad , \quad I \longrightarrow X^\bullet$$

i.e. for each $k \in I$, a map

$$X^i \longrightarrow \operatorname*{colim}_j \mathrm{Map}_{\mathcal{C}}(X^j, X^k) \quad \text{for some } i$$

A pro-monoid (Defn. 2) is also a pro-monoid (Defn. 3).

II. Motivation/Application

$\mathcal{C}$: symmetric sequences in sym. mon. cat $(\mathcal{A}, \otimes, 1)$

↳ $(A_1, A_2, \dots)$ with Σ_n-action on A_n

$\circ$: composition product

$$(A \circ B)_n = \coprod_{\substack{\text{partitions of} \\ \{1,\dots,n\}}} A_\ell \otimes B_{n_1} \otimes \cdots \otimes B_{n_\ell}$$

I : unit sym. seq.

$$I_\ell = \begin{cases} 1 & \text{if } \ell = 1 \\ \emptyset & \text{if } \ell > 1 \end{cases}$$

Then a monoid in $(\mathcal{C}, \circ, I)$ is an <u>operad</u> in $\mathcal{A}$.

If $\mathcal{A}$ is closed sym. mon. then $\circ$ is right-closed:

$$\mathrm{Map}^{\circ}(A,B)_\ell = \prod_n \Bigg[\prod_{\substack{\text{part. of} \\ \{1,\dots,n\}}} \mathrm{Map}_{\mathcal{A}}\left(A_{n_1}\otimes\cdots\otimes A_{n_\ell},\, B_\ell\right)\Bigg]^{\Sigma_n}$$

<u>Defn.</u> [Blom–C.]

A <u>pro-operad</u> in $\mathcal{A}$ is a pro-monoid (Defn. 3) in $(\mathcal{C},\circ,I)$, i.e. a pro-symmetric sequence $A^\bullet$ with, for each $k\in\mathbb{I}$, maps

$$A^i_\ell \longrightarrow \operatorname*{colim}_j \mathrm{Map}_{\mathcal{A}}\left(A^j_{n_1}\otimes\cdots\otimes A^j_{n_\ell},\, A^k_n\right) \quad \text{for some } i\in\mathbb{I}.$$

Theorem [Blom–C.]

For any (differentiable) ∞-category $\mathcal{C}$, there is a pro-(∞-)operad $\partial^{\bullet}_{\mathcal{C}}$ in spectra s.t.

$$\partial_* : \mathrm{Poly}(\mathcal{C}, \mathrm{Sp}) \xrightarrow{\sim} \mathrm{RMod}_{<\infty}(\partial^{\bullet}_{\mathcal{C}})$$

[Goodwillie]

Remark Sometimes $\partial^{\bullet}_{\mathcal{C}}$ is a pro-operad (Defn. 1) i.e. a cofiltered diagram of operads

III Construction of closed structure on $\mathrm{Pro}(\mathcal{C})$

For closed category $\mathcal{C}$, there are notions of $\mathcal{C}$-enriched category, functor, nat. trans. and fully faithful embeddings

$$\mathcal{C} \hookrightarrow \mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C})^{op} \xhookrightarrow{\text{yoneda}} \mathrm{Fun}(\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C}), \mathrm{Set})$$

$$X \longmapsto \mathrm{Map}_{\mathcal{C}}(X, -)$$

[Eilenberg-Kelly, LaPlaza]

If $\mathcal{C}$ has filtered colimits, then $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C})^{op}$ has cofiltered limits, so we get a canonical extension

$$\begin{aligned} \mathrm{Pro}(\mathcal{C}) &\longrightarrow \mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C})^{op} \\ X^{\bullet} &\longmapsto \operatorname{colim}_i \mathrm{Map}_{\mathcal{C}}(X^i,-) \end{aligned}$$

which is still fully faithful: for $X^{\bullet}, Y^{\bullet} \in \mathrm{Pro}(\mathcal{C})$:

$$\mathrm{Nat}_{\mathcal{C}}\left(\operatorname{colim}_j \mathrm{Map}_{\mathcal{C}}(Y^j,-), \operatorname{colim}_i \mathrm{Map}_{\mathcal{C}}(X^i,-)\right)$$

$$= \lim_j \mathrm{Nat}_{\mathcal{C}}\left(\mathrm{Map}_{\mathcal{C}}(Y^j,-), \operatorname{colim}_i \mathrm{Map}_{\mathcal{C}}(X^i,-)\right) = \lim_j \operatorname{colim}_i \mathrm{Hom}_{\mathcal{C}}(X^i,Y^j)$$

So we have an embedding

$$\mathrm{Pro}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C}), \mathrm{Set})$$

and the closed structure on $\mathrm{Fun}(\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C}), \mathrm{Set})$ restricts to the required closed structure on $\mathrm{Pro}(\mathcal{C})$. ∎

Remark

Monoids in $\mathrm{Pro}(\mathcal{C})$ are monoids in $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C})^{op}$

(pro-monoids (Defn. 3)) ($\mathcal{C}$-enriched comonads on $\mathcal{C}$)

Defn A module over a pro-monoid X° is a coalgebra over the corresponding comonad.