Homotopical recognition of diagram categories

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CT 2025, Brno
July 16, 2025
Joint work with David White

Outline

Recognition of presheaf categories: classical results

Ordinary case:

Theorem (M. Bunge '69 – PhD thesis)

$\mathscr{E}$ is isomorphic to a presheaf category iff it is a cocomplete atomic regular category with a generating set $\mathscr{A}$ of atoms.

$$\mathscr{E} \simeq \mathrm{Set}^{\mathscr{A}^{\mathrm{op}}}$$

$A \in \mathscr{A}$ is an atom if $\hom(A, -)$ commutes with all colimits.

Simplicial case:

Theorem (W.G. Dwyer and D. Kan '84)

Let $\mathscr{M}$ be a simplicial category equipped with a set of orbits $\mathscr{O}$. Then there exists a model structure on $\mathscr{M}$ Quillen equivalent to $\mathcal{P}(\mathscr{O})$.

$$\mathscr{M} \simeq_Q \mathcal{S}^{\mathscr{O}^{\mathrm{op}}}$$

$O \in \mathscr{O}$ is an orbit if $\hom(O, -)$ commutes with cellular constructions up to homotopy.

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Bunge's conditions:
$\mathscr{E}$ cocomplete, atomic, regular

cocomplete: $\mathscr{E}$ is closed under arbitrary colimits;

atomic: $\mathscr{E}$ has a dense subcategory $\mathscr{A}$ of universally presentable objects;

dense: every object is canonically a colimit of objects in $\mathscr{A}$.

regular: $\mathscr{E}$ has finite limits, kernel pairs, and satisfies exactness conditions like existence of images and the image factorization (regular epi-mono).

Dwyer-Kan orbits: $\mathscr{M}$ is equipped with a set of orbits $\{O_e\}_{e\in E}$ if

Q0: $\mathscr{M}$ is closed under arbitrary limits and colimits;

Q1: $\forall e \in E$,

$$\begin{array}{ccc} (O_{e'} \otimes K)^{O_e} & \longrightarrow & X_a^{O_e} \\ \big\downarrow & \text{hom. p.-o.} & \big\downarrow \\ (O_{e'} \otimes L)^{O_e} & \longrightarrow & X_{a+1}^{O_e}, \end{array}$$

where $(K \hookrightarrow L) \in \mathcal{S}_{\text{fin}}$;

Q2: $\forall \alpha \forall e \in E$, $(\operatorname{colim}_{a<\alpha} X_a)^{O_e} \simeq \operatorname{colim}_{a<\alpha} X_a^{O_e}$;

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Theorem: (Dwyer-Kan, '84)

Theorem

Let $\mathscr{M}$ be a simplicial category equipped with a set of orbits $\mathcal{O} = \{O_e\}_{e\in E}$. Then $\mathscr{M}$ is a model category with $f\colon X \to Y$ a W.E. or a fib if the induced map $\mathrm{hom}(O_e, f)\colon \mathrm{hom}(O_e, X) \to \mathrm{hom}(O_e, Y)$ is a W.E. or a fib., respectively. Moreover, the adjunction

$$\mathscr{M} \underset{\mathrm{hom}(\mathcal{O},-)}{\overset{-\otimes_{\mathcal{O}}\mathrm{Inc}}{\rightleftarrows}} \mathcal{S}^{\mathcal{O}^{\mathrm{op}}}, \qquad \mathrm{Inc}\colon \mathcal{O} \hookrightarrow \mathscr{M}$$

is a Quillen equivalence if the category of presheaves is equipped with the projective model structure.

Examples of Dwyer-Kan orbits

- Bredon homotopy theory: $\mathscr{M} = \mathcal{S}^G$, $\mathcal{O}_G = \{G/H \,|\, H < G\}$

$$\mathcal{S}^G \underset{\text{fixed points}}{\overset{\text{Elmendorf}}{\rightleftarrows}} \mathcal{S}^{\mathcal{O}_G^{op}},$$

- Relative homotopy theory: Balmer-Matthey (2004) $\mathscr{M} = \mathcal{S}^{\mathscr{D}}$, $\mathscr{C} \subset \mathscr{D}$, $\mathcal{O} = \{R^C = \hom(C, -) \,|\, C \in \mathscr{C}\} \simeq \mathscr{C}^{op}$

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- Farjoun-Zabrodsky orbits (1986): $\mathscr{M} = \mathcal{S}^D$, even if D is small, $\mathcal{O}_D = \{\underset{\sim}{T} \,|\, \mathrm{colim}_D \underset{\sim}{T} = *\}$ may be large, and $\mathscr{M}$ not cofibrantly generated.

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Homotopy Atoms

Let $\mathscr{M}$ be a $\mathscr{V}$-model category, $\mathscr{V}$ combinatorial, generated by $I_{\mathscr{V}} = \{A_i \hookrightarrow B_i \mid i \in I\}$.

Definition

$\mathscr{M}$ is equipped with a set of *homotopy atoms* if there exists a set of cofibrant objects $\mathcal{H} \subset \mathscr{M}$ such that

1. The functors $\{\hom(T, -) \mid T \in \mathcal{H}\}$ jointly reflect weak equivalences between fibrant objects;
2. The functors $\{\hom(T, \widehat{-}) \mid T \in \mathcal{H}\}$ commute with homotopy pushouts, sequential homotopy colimits, and $- \otimes A_i$ and $- \otimes B_i$, up to weak equivalence.

Related work: Anna Montarulli, 'Representation theorems for abelian and model categories'. 2023
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Main Theorem

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Let $\mathscr{M}$ be a $\mathscr{V}$-model category. There exists a small $\mathscr{V}$-category $\mathscr{C}$ and a Quillen equivalence $R\colon \mathscr{M} \rightleftarrows \mathscr{V}^{\mathscr{C}^{op}} : L$ iff $\mathscr{M}$ may be equipped with a set of homotopy atoms.

proof idea:
($\Rightarrow$) define $\mathcal{H} = \{T_C = L(\hom(-, C)) \mid C \in \mathscr{C}\}$. Note $RM(C) = \hom(T_C, M)$ by Yoneda.
Representable functors in proj. model str. are cofibrant.
Yoga of weighted homotopy colimits to check homotopy atoms.
($\Leftarrow$) proof idea: Let $\mathscr{C}$ be full $\mathscr{V}$-subcat. of $\mathscr{M}$ on objects $\mathcal{H}$. Let $RM(T) = \hom(T, M)$ and $L(-) = (-) \otimes_{\mathscr{C}} H$ where $H : \mathscr{C} \hookrightarrow \mathscr{M}$.
Prove Q.E. using cellular induction.

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Example 2 (Schwede-Shipley '03): Let $\mathscr{M}$ be a stable simplicial model category equipped with a set of (cofibrant) compact generators $\mathcal{G}$, then the spectral category $\mathrm{Sp}^{\Sigma}(\mathscr{M})$ is also equipped with a set of compact generators $\Sigma^{\infty}\mathcal{G}$ and it is Quillen equivalent to the category of modules over a 'ring with several objects' $\mathcal{E} = \mathrm{End}(\mathcal{G}_{\mathrm{fib}})$

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Example 4: Sarah Yeakel's isovariant homotopy theory + isovariant Elmendorf theorem (equivariant maps $f : X \to Y$ plus equality of stabilizers $G_x = G_{f(x)}$).

Example 5: Gu's model structures on diagrams of categories, with orbit model structures.

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Categories of functors (not necessarily presheaves)

Next goal: Learn to recognize functor categories of the form $\mathscr{N}^D$, where $\mathscr{N}$ is a combinatorial $\mathscr{V}$-category and D is a small $\mathscr{V}$-category.
Suppose $\mathscr{E} \subset \mathscr{N}$ is a full subcategory such that $\mathscr{N}$ is Quillen equivalent to the localization of $\mathscr{V}^{\mathscr{E}^{\mathrm{op}}}$ w.r.t. a set of maps $\mathcal{F}$. A $\mathscr{V}$-category $\mathscr{M}$ is equipped with a natural function complex in $\mathscr{N}$ if $\mathscr{M}$ is a $\mathscr{V}^{\mathscr{E}^{\mathrm{op}}}$-category and $\forall f \in \mathcal{F}\ M \in \mathscr{M},\ f \otimes \tilde{M}$ is a w.eq.

Example: let $\mathscr{V} = \mathcal{S}_*$, and Sp = Bousfield-Friedlander spectra = $\mathcal{S}_*^{Sph}$ where $\hom_{\mathrm{Sph}}(i,j) = S^{j-i}$ if $i \le j$ and $*$ otherwise. Let $\mathscr{E} = \{\Sigma^{-i}(\Sigma^\infty S^0) \mid i \ge 0\}$, and $\mathcal{S}_*^{\mathscr{E}^{\mathrm{op}}} \leftrightarrows$ Sp is a Quillen pair that becomes a Quillen equivalence after a left Bousfield localization that turns homotopy pullbacks into homotopy pushouts. Then Sp is equipped with a natural function complex over Sp and so are categories of diagrams of spectra.

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Recognition of functor categories

Suppose $\mathscr{M}$ is equipped with a natural function complex in $\mathscr{N} = (\mathscr{V}^{\mathscr{E}^{op}})_{\mathcal{F}}$ and $I_{\mathscr{N}} = \{A_i \hookrightarrow B_i \mid i \in I\}$ is a set of generating cofibrations. Suppose in addition that there is a full subcategory $\mathscr{F} \subset \mathscr{N}$ such that

1. The functors $\{\mathrm{Nat}(F, -) \mid F \in \mathscr{F}\}$ jointly reflect weak equivalences of fibrant objects;
2. The functors $\{\mathrm{Nat}(F, \widehat{-}) \mid F \in \mathscr{F}\}$ commute with homotopy pushouts, sequential homotopy colimits, and $- \otimes A_i$ and $- \otimes B_i$, up to weak equivalence.

The objects of $\mathscr{F}$ are called homotopy $\mathscr{N}$-atoms.

Theorem

$\mathscr{M}$ is equipped with a set of homotopy $\mathscr{N}$-atoms $\mathscr{F}$ if and only if $\mathscr{M}$ is Quillen equivalent to the diagram category $\mathscr{N}^{\mathscr{F}^{op}}$.

Classification of (finitary) polynomial functors I

n-excisive = 'polynomial of degree $\leq n$': takes strongly cocartesian $(n+1)$-cubes to cartesian.

Notation: $\mathscr{V} = \mathcal{S}_*$, $\mathscr{M} = (\mathrm{Sp}^{\mathcal{S}_*^{\mathrm{fin}}})_{n\text{-exc}}$; $R^{S^0}(-) = \hom(S^0, -)$, $\mathscr{F} = \{\Sigma^\infty(\bigwedge_{i=1}^k R^{S^0})_{\mathrm{cof}}\}_{k=1}^n$.

Lemma (Biedermann-Ch.-Röndigs, '07)

For homotopy functor $F \in \mathrm{Sp}^{\mathcal{S}_*^{\mathrm{fin}}}$, the n-th cross-efffect may be computed as $\mathrm{Nat}\left(\Sigma^\infty(\bigwedge_{i=1}^n R^{S^0})_{\mathrm{cof}}, F\right) = cr_n(S^0, \ldots, S^0)$.

Goodwillie: $\forall F, G \in \mathrm{Sp}^{\mathcal{S}_*}$ n-homogeneous, if $f\colon F \to G$ is such that $cr_n(f)$ is a weak equivalence, then f is a weak equivalence. Inductive argument, using Goodwillie's delooping theorem shows that $\{\mathrm{Nat}(F, -) \mid F \in \mathscr{F}\}$ jointly reflect weak equivalences of n-excisive functors.

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Classification of (finitary) polynomial functors II

Theorem

$(\mathrm{Sp}^{\mathcal{S}_*^{\mathrm{fin}}})_{n\text{-exc}}$ is Quillen equivalent to the projective model structure on $\mathrm{Sp}^{\mathscr{F}^{\mathrm{op}}}$.

Notation: Let $\Omega_{\leq n}$ be the category of finite non-empty sets with surjections as morphisms, and $\Omega^+_{\leq n}$ is the category on the same objects and $\Omega^+_{\leq n}(m,k) = \Omega_{\leq n}(m,k)^+$

Previous results: Dwyer-Rezk(unpublished), Arone-Ching ('16) There is a Quillen equivalence $\mathrm{Sp}^{\Omega_{\leq n}} \leftrightarrows (\mathrm{Sp}^{\mathcal{S}_*^{\mathrm{fin}}})_{n\text{-exc}}$.

Comparison of the results: There is a Dwyer-Kan equivalence of categories $\Omega_{\leq n}(m,k)^+ \to \mathscr{F}^{\mathrm{op}}$, hence the equivalence of the ∞-categories of functors. But the underlying categories of $\mathrm{Sp}^{\Omega^+_{\leq n}} \in \mathcal{S}$-cat. and $\mathrm{Sp}^{\Omega_{\leq n}} \in \mathcal{S}_*$-cat. are isomorphic, hence carry the same model structure.

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