A NEW FRAMEWORK FOR LIMITS IN DOUBLE CATEGORIES

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International Category Theory conference
Masaryk University, Brno, Czechia
15 July 2025

MOTIVATION & OVERVIEW

double categories $\geqslant$ 2-categories + bicategories

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- Key advantage: double categories have 2 kinds of objects:

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\mathbb{D} \; = & D_0 \arrow[r, "\mathrm{id}" description] & D_1 \arrow[l, shift right=3, "\mathrm{dom}"'] \arrow[l, shift left=3, "\mathrm{cod}"] & D_1 \times_{D_0} D_1 \arrow[l, shift right=3, "\pi_1"'] \arrow[l, "\odot" description] \arrow[l, shift left=3, "\pi_2"] \\
 & \text{objects} \arrow[u, squiggly] & \text{loose morphisms} \arrow[u, squiggly] &
\end{tikzcd}$$

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PART 1: Limits indexed by double categories $\mathbb{I}$

$$\begin{tikzcd} & \lim F \arrow[dl, "\gamma_A"'] \arrow[dr, "\gamma_B"] & \\ FA \arrow[rr, "Ff"'] & & FB \end{tikzcd} \qquad \begin{tikzcd} \lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\gamma_p"] & \lim F \arrow[d, "\gamma_D"] \\ FC \arrow[r, "Fp"', "\shortmid"{marking}] & FD \end{tikzcd}$$

- Introduced by Grandis-Paré in 1999.
- Constructed from limits in D_0 and tabulators.

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PART 2: Limits indexed by loose distributors $\mathbb{I} \overset{P}{\longrightarrow\!\!\!\!\!\!\shortmid\!\!\shortmid}\ \mathbb{J}$

$$\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_q"] & \lim G \arrow[d, "\psi_X"] \\
FA \arrow[r, "\Phi q"', "\shortmid"{marking}] & GX
\end{tikzcd}$$

- Capture parallel limits and many new examples!
- Main theorem: characterising $\mathbb{D}$ which admit all limits.

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left unitor   right unitor   associator

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- Category $\mathcal{C}$ $\rightsquigarrow$ double category $\mathbb{T}\mathrm{i}(\mathcal{C})$
  - * objects & tight morphisms come from $\mathcal{C}$
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A \times C \arrow[rr, bend left, "p"] \arrow[d, "f \times g"'] & \Downarrow & \{\bot \to \top\} \\
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- **$\mathbb{D}$ist** – categories, functors, distributors/profunctors

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A unitary lax functor $F : \mathbb{C} \longrightarrow \mathbb{D}$ is an assignment

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preserving tight identities & composites and loose identities, together with laxator cells:

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$$\mathbb{T}\mathrm{i}(\mathcal{C}) \longrightarrow \mathbb{D} \quad \leftrightsquigarrow \quad \mathcal{C} \longrightarrow \mathbb{D}_0$$

unitary lax functor　　　　functor

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FA \arrow[rr, "F(p \odot q)"', "\shortmid"{marking}] & & FC
\end{tikzcd}$$

$$\mathbb{T}_i(\mathcal{C}) \longrightarrow \mathbb{D} \quad \leftrightsquigarrow \quad \mathcal{C} \longrightarrow \mathcal{D}_0$$

unitary lax functor

functor

A limit of $F : \mathbb{I} \longrightarrow \mathbb{D}$ is an object $\lim F$ and a terminal cone γ

LIMITS INDEXED BY DOUBLE CATEGORIES

A unitary lax functor $F:\mathbb{C}\longrightarrow\mathbb{D}$ is an assignment

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] & & FA \arrow[r, "Fp", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fg"] \\
B \arrow[r, "q"', "\shortmid"{marking}] & D \arrow[r, phantom, "\rightsquigarrow"] & & FB \arrow[r, "Fq"', "\shortmid"{marking}] & FD
\end{tikzcd}$$

preserving tight identities & composites and loose identities, together with laxator cells:

$$\begin{tikzcd}
FA \arrow[r, "Fp", "\shortmid"{marking}] \arrow[d, equal] & FB \arrow[r, "Fq", "\shortmid"{marking}] \arrow[d, phantom, "\underline{c}(p,q)"] & FC \arrow[d, equal] \\
FA \arrow[rr, "F(p\odot q)"', "\shortmid"{marking}] & & FC
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{T}\mathrm{i}(\mathbb{C}) \arrow[r] & \mathbb{D} \arrow[r, phantom, "\leftrightsquigarrow"] & C \arrow[r] & D_0
\end{tikzcd}$$

unitary lax functor

functor

A limit of $F:\mathbb{I}\longrightarrow\mathbb{D}$ is an object $\lim F$ and a terminal cone γ which provides for each $f:A\rightarrow B$ and $p:C\nrightarrow D$

$$\begin{tikzcd}
FA \arrow[r, "Ff"'] & FB & & FC \arrow[r, "Fp"', "\shortmid"{marking}] & FD
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B \arrow[r, "q"', "\shortmid" marking] & D \arrow[r, phantom, "\rightsquigarrow"] & {} & FB \arrow[r, "Fq"', "\shortmid" marking] & FD
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FA \arrow[rr, "F(p \odot q)"', "\shortmid" marking] & & FC
\end{tikzcd}$$

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$$\begin{tikzcd}
& \lim F \arrow[dl, "\gamma_A"'] \arrow[dr, "\gamma_B"] & & \lim F \arrow[r, "id", "\shortmid"{marking}] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\gamma_p"] & \lim F \arrow[d, "\gamma_D"] \\
FA \arrow[rr, "Ff"'] & & FB & FC \arrow[r, "Fp"', "\shortmid"{marking}] & FD
\end{tikzcd}$$

natural w.r.t. cells in $\mathbb{I}$

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B \arrow[r, "q"', "\shortmid"{marking}] & D
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
FA \arrow[r, "Fp", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & FC \arrow[d, "Fg"] \\
FB \arrow[r, "Fq"', "\shortmid"{marking}] & FD
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FA \arrow[rr, "Ff"'] & & FB
\end{tikzcd}
\qquad
\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\gamma_p"] & \lim F \arrow[d, "\gamma_D"] \\
FC \arrow[r, "Fp"', "\shortmid"{marking}] & FD
\end{tikzcd}$$

natural w.r.t. cells in $\mathbb{I}$ such that $\gamma_{\mathrm{id}_A} = \mathrm{id}_{\gamma_A}$

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B \arrow[r, "q"', "\shortmid"{marking}] & D \arrow[r, phantom, "\rightsquigarrow"] & & FB \arrow[r, "Fq"', "\shortmid"{marking}] & FD
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FA \arrow[rr, "F(p \odot q)"', "\shortmid"{marking}] & & FC
\end{tikzcd}$$

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\mathbb{T}\mathrm{i}(\mathcal{C}) \arrow[r] & \mathbb{D} & \leftrightsquigarrow & \mathcal{C} \arrow[r] & \mathbb{D}_0
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A limit of $F: \mathbb{I} \to \mathbb{D}$ is an object $\lim F$ and a terminal cone γ which provides for each $f: A \to B$ and $p: C \mathrel{\mapstochar\rightarrow} D$

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FA \arrow[rr, "Ff"'] & & FB & & FC \arrow[r, "Fp"', "\shortmid"{marking}] & FD
\end{tikzcd}$$

natural w.r.t. cells in $\mathbb{I}$ such that $\gamma_{\mathrm{id}_A} = \mathrm{id}_{\gamma_A}$ and

$$\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\gamma_p"] & \lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_B"] \arrow[dr, phantom, "\gamma_q"] & \lim F \arrow[d, "\gamma_C"] & & \lim F \arrow[r, "\mathrm{id} \odot \mathrm{id}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim F \arrow[d, equal] \\
FA \arrow[r, "Fp", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\underline{c}(p,q)"] & FB \arrow[r, "Fq", "\shortmid"{marking}] & FC \arrow[d, equal] & = & \lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\gamma_{p \odot q}"] & \lim F \arrow[d, "\gamma_C"] \\
FA \arrow[rr, "F(p \odot q)"', "\shortmid"{marking}] & & FC & & FA \arrow[r, "F(p \odot q)"', "\shortmid"{marking}] & FC
\end{tikzcd}$$

TABULATORS & TIGHT LIMITS

- A tabulator is a limit whose shape is

$$\mathbb{2} = \{0 \relbar\joinrel\mapstochar\joinrel\rightarrow 1\}$$

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- The tabulator of a loose morphism $p : A \nrightarrow B$ is a cone

$$\begin{tikzcd}
T_p \arrow[r, "id", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & T_p \arrow[d, "\pi_B"] \\
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\end{tikzcd}$$

- A double category $\mathbb{D}$ admits all tabulators if and only if the functor $\mathrm{id} : D_0 \longrightarrow D_1$ has a right adjoint.

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A \arrow[r, "\shortmid"{marking}, "p"'] & B
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- A double category $\mathbb{D}$ admits all tabulators if and only if the functor $id : D_0 \longrightarrow D_1$ has a right adjoint.

- In $\mathbb{S}pan$, $T(A \leftarrow P \rightarrow B) = P$

- In $\mathbb{R}el$, $T(R : A \times B \longrightarrow \{\bot \to \top\}) = \{(a,b) \in A \times B \mid R(a,b) = \top\}$

- In $\mathbb{D}ist$, the category of elements of $P : A^{op} \times B \longrightarrow Set$.

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$$\begin{tikzcd}
Tp \arrow[r, "\shortmid" marking, "\mathrm{id}"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & Tp \arrow[d, "\pi_B"] \\
A \arrow[r, "\shortmid" marking, "p"'] & B
\end{tikzcd}$$

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- Tight limits in a double category $\mathbb{D}$ are precisely limits in the underlying category D_0 of objects and tight morphisms.

$$\mathbb{T}\mathrm{i}(C) \xrightarrow{F} \mathbb{D} \qquad \leftrightsquigarrow \qquad C \xrightarrow{F_0} D_0$$

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$$\mathbb{T}\mathrm{i}(\mathcal{C}) \xrightarrow{F} \mathbb{D} \qquad \leftrightsquigarrow \qquad \mathcal{C} \xrightarrow{F_0} \mathbb{D}_0$$

- Examples: products, equalisers, pullbacks, terminal objects.

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- Examples: products, equalisers, pullbacks, terminal objects.

Theorem (Grandis-Paré, 99)

A double category $\mathbb{D}$ admits limits indexed by any double category $\mathbb{I}$ <u>if and only if</u> $\mathbb{D}$ admits tight limits and tabulators.

LOOSE DISTRIBUTORS & ALTERATIONS

A loose distributor $P : \mathbb{C} \nrightarrow \mathbb{D}$ is

1) a distributor between pseudo category objects in CAT

$$C_0 \xleftarrow{s} P \xrightarrow{t} D_0 \qquad C_1 \times_{C_0} P \xrightarrow{\triangleright} P \qquad P \times_{D_0} D_1 \xrightarrow{\triangleleft} P$$

2)

3)

LOOSE DISTRIBUTORS & ALTERATIONS

A **loose distributor** $P\colon \mathbb{C} \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} \mathbb{D}$ is

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$$C_0 \xleftarrow{s} P \xrightarrow{t} D_0 \qquad C_1 \times_{C_0} P \xrightarrow{\triangleright} P \qquad P \times_{D_0} D_1 \xrightarrow{\triangleleft} P$$

2) a functor $\mathbb{P} \longrightarrow \mathbb{2}$ into free double category $\{0 \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} 1\}$

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2) a functor $\mathbb{P} \longrightarrow \mathbb{2}$ into free double category $\{0 \nrightarrow 1\}$

3) for each $A \in \mathbb{C}$ and $X \in \mathbb{D}$, a collection $P(A, X)$ of loose heteromorphisms $A \nrightarrow X$,

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2) a functor $\mathbb{P} \longrightarrow \mathbb{2}$ into free double category $\{0 \nrightarrow 1\}$

3) for each $A \in \mathbb{C}$ and $X \in \mathbb{D}$, a collection $P(A, X)$ of **loose heteromorphisms** $A \nrightarrow X$, and set of **heterocells**

$$\begin{tikzcd}
\mathbb{C} & A \arrow[d, "f"'] & & A \arrow[r, "p", "|" marking] \arrow[d, "f"'] & X \arrow[d, "g"] & & X \arrow[d, "g"] & \mathbb{D} \\
& B & & B \arrow[r, "q"', "|" marking] & Y & & Y &
\end{tikzcd}$$

for each frame of tight morphisms and loose heteromorphisms

LOOSE DISTRIBUTORS & ALTERATIONS

A loose distributor $P : \mathbb{C} \mathrel{\relbar\mkern-4mu\mapstochar\mkern-2mu\mapstochar\mkern-2mu\rightarrow} \mathbb{D}$ is

1) a distributor between pseudo category objects in CAT

$$C_0 \xleftarrow{s} P \xrightarrow{t} D_0 \qquad C_1 \times_{C_0} P \xrightarrow{\triangleright} P \qquad P \times_{D_0} D_1 \xrightarrow{\triangleleft} P$$

2) a functor $\mathbb{P} \longrightarrow \mathbb{2}$ into free double category $\{0 \mathrel{\relbar\mkern-4mu\mapstochar\mkern-2mu\rightarrow} 1\}$

3) for each $A \in \mathbb{C}$ and $X \in \mathbb{D}$, a collection $P(A, X)$ of loose heteromorphisms $A \mathrel{\relbar\mkern-4mu\mapstochar\mkern-2mu\rightarrow} X$, and set of heterocells

$$\begin{tikzcd} \mathbb{C} & A \arrow[d, "f"'] & & A \arrow[r, "p", "|" marking] \arrow[d, "f"'] & X \arrow[d, "g"] & & X \arrow[d, "g"] & \mathbb{D} \\ & B & & B \arrow[r, "q"', "|" marking] & Y & & Y & \end{tikzcd}$$

for each frame of tight morphisms and loose heteromorphisms together with compatible left action by $\mathbb{C}$ and right action by $\mathbb{D}$.

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1) a distributor between pseudo category objects in CAT

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2) a functor $\mathbb{P} \longrightarrow \mathbb{2}$ into free double category $\{0 \mathrel{\mapstochar\rightarrow} 1\}$

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\mathbb{C} & A \arrow[d, "f"'] & & A \arrow[r, "p", "\shortmid" marking] \arrow[d, "f"'] & X \arrow[d, "g"] & & X \arrow[d, "g"] & \mathbb{D} \\
 & B & & B \arrow[r, "q"', "\shortmid" marking] & Y & & Y &
\end{tikzcd}$$

for each frame of tight morphisms and loose heteromorphisms together with compatible left action by $\mathbb{C}$ and right action by $\mathbb{D}$.

collage of
$P : \mathbb{C} \mathrel{\mapstochar\mapstochar\rightarrow} \mathbb{D}$

$$\mathbb{C} \longrightarrow \mathbb{P} \longleftarrow \mathbb{D}$$

LOOSE DISTRIBUTORS & ALTERATIONS

A loose distributor $P : \mathbb{C} \mathrel{-\!\!\shortmid\!\shortmid\!\!\!\to} \mathbb{D}$ is

1) a distributor between pseudo category objects in CAT

$$C_0 \xleftarrow{s} P \xrightarrow{t} D_0 \qquad C_1 \times_{C_0} P \xrightarrow{\triangleright} P \qquad P \times_{D_0} D_1 \xrightarrow{\triangleleft} P$$

2) a functor $\mathbb{P} \longrightarrow \mathbb{2}$ into free double category $\{0 \mathrel{-\!\!\shortmid\!\shortmid\!\!\!\to} 1\}$

3) for each $A \in \mathbb{C}$ and $X \in \mathbb{D}$, a collection $P(A, X)$ of loose heteromorphisms $A \nrightarrow X$, and set of heterocells

$$\begin{tikzcd}
 & A \arrow[dd, "f"'] & & A \arrow[r, "p", "\shortmid" marking] \arrow[dd, "f"'] & X \arrow[dd, "g"] & & X \arrow[dd, "g"] & \\
\mathbb{C} & & & & & & & \mathbb{D} \\
 & B & & B \arrow[r, "q"', "\shortmid" marking] & Y & & Y &
\end{tikzcd}$$

for each frame of tight morphisms and loose heteromorphisms together with compatible left action by $\mathbb{C}$ and right action by $\mathbb{D}$.

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & \mathbb{P} & \mathbb{D} \arrow[l]
\end{tikzcd}$$

collage of $P : \mathbb{C} \mathrel{-\!\!\shortmid\!\shortmid\!\!\!\to} \mathbb{D}$

An (unitary lax) alteration with frame

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "P", "\shortmid\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid\shortmid" marking] & \mathbb{D}
\end{tikzcd}$$

F, G unitary lax functors

LOOSE DISTRIBUTORS & ALTERATIONS

A **loose distributor** $P : \mathbb{C} \xrightarrow{\shortmid\shortmid} \mathbb{D}$ is

1) a distributor between pseudo category objects in CAT

$$C_0 \xleftarrow{s} P \xrightarrow{t} D_0 \qquad C_1 \times_{C_0} P \xrightarrow{\triangleright} P \qquad P \times_{D_0} D_1 \xrightarrow{\triangleleft} P$$

2) a functor $\mathbb{P} \longrightarrow \mathbb{2}$ into free double category $\{0 \xrightarrow{\shortmid} 1\}$

3) for each $A \in \mathbb{C}$ and $X \in \mathbb{D}$, a collection $P(A,X)$ of **loose heteromorphisms** $A \xrightarrow{\shortmid} X$, and set of **heterocells**

$$\begin{tikzcd}
\mathbb{C} & A \arrow[d, "f"'] & & A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] & X \arrow[d, "g"] & & X \arrow[d, "g"] & \mathbb{D} \\
& B & & B \arrow[r, "q"', "\shortmid"{marking}] & Y & & Y &
\end{tikzcd}$$

for each frame of tight morphisms and loose heteromorphisms together with compatible **left action by $\mathbb{C}$** and **right action by $\mathbb{D}$**.

collage of
$P : \mathbb{C} \xrightarrow{\shortmid\shortmid} \mathbb{D}$

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & \mathbb{P} & \mathbb{D} \arrow[l]
\end{tikzcd}$$

An (unitary lax) **alteration** with frame

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "P", "\shortmid\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid\shortmid"{marking}] & \mathbb{D}
\end{tikzcd}$$

F, G unitary lax functors

is an assignment on heteromorphisms and heterocells

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "g"] & & FA \arrow[r, "\Phi p", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Phi\alpha"] & GX \arrow[d, "Gg"] \\
B \arrow[r, "q"', "\shortmid"{marking}] & Y & \longmapsto & FB \arrow[r, "\Phi q"', "\shortmid"{marking}] & GY
\end{tikzcd}$$

in $\mathbb{D}$

LOOSE DISTRIBUTORS & ALTERATIONS

A **loose distributor** $P : \mathbb{C} \relbar\joinrel\mapstochar\mkern-4mu\mapstochar\joinrel\rightarrow \mathbb{D}$ is

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$$C_0 \xleftarrow{s} P \xrightarrow{t} D_0 \qquad C_1 \times_{C_0} P \xrightarrow{\triangleright} P \qquad P \times_{D_0} D_1 \xrightarrow{\triangleleft} P$$

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$$\begin{tikzcd}
\mathbb{C} & A \arrow[d, "f"'] & & A \arrow[r, "p", "\shortmid" marking] \arrow[d, "f"'] & X \arrow[d, "g"] & & X \arrow[d, "g"] & \mathbb{D} \\
 & B & & B \arrow[r, "q"', "\shortmid" marking] & Y & & Y &
\end{tikzcd}$$

for each frame of tight morphisms and loose heteromorphisms together with compatible **left action by $\mathbb{C}$** and **right action by $\mathbb{D}$**.

collage of
$P : \mathbb{C} \relbar\joinrel\mapstochar\mkern-4mu\mapstochar\joinrel\rightarrow \mathbb{D}$

$$\begin{tikzcd}
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\end{tikzcd}$$

An (unitary lax) **alteration** with frame

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "P", "\shortmid\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid\shortmid" marking] & \mathbb{D}
\end{tikzcd}$$

F, G unitary lax functors

is an assignment on heteromorphisms and heterocells

$$\begin{tikzcd}
A \arrow[r, "p", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "g"] & & FA \arrow[r, "\Phi p", "\shortmid" marking] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Phi\alpha"] & GX \arrow[d, "Gg"] \\
B \arrow[r, "q"', "\shortmid" marking] & Y & \longmapsto & FB \arrow[r, "\Phi q"', "\shortmid" marking] & GY
\end{tikzcd} \quad \text{in } \mathbb{D}$$

preserving identity and composite heterocells together with cells

$$\begin{tikzcd}
FA' \arrow[r, "Fq", "\shortmid" marking] \arrow[d, equal] & FA \arrow[r, "\Phi p", "\shortmid" marking] \arrow[d, phantom, "\underline{c}^{\triangleright}(q{,}p)"] & GX \arrow[d, equal] & FA \arrow[r, "\Phi p", "\shortmid" marking] \arrow[d, equal] & GX \arrow[r, "Gr", "\shortmid" marking] \arrow[d, phantom, "\underline{c}^{\triangleleft}(p{,}r)"] & GX' \arrow[d, equal] \\
FA' \arrow[rr, "\Phi(q \triangleright p)"', "\shortmid" marking] & & GX & FA \arrow[rr, "\Phi(p \triangleleft r)"', "\shortmid" marking] & & GX'
\end{tikzcd}$$

LIMITS INDEXED BY LOOSE DISTRIBUTORS

Suppose $(\lim F, \gamma)$ and $(\lim G, \psi)$ are limits of $F : \mathbb{I} \longrightarrow \mathbb{D}$ and $G : \mathbb{J} \longrightarrow \mathbb{D}$, respectively.

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Suppose $(\lim F, \gamma)$ and $(\lim G, \psi)$ are limits of $F : \mathbb{I} \longrightarrow \mathbb{D}$ and $G : \mathbb{J} \longrightarrow \mathbb{D}$, respectively. The limit of an alteration

$$\begin{tikzcd} \mathbb{I} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\ \mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid"{marking}] & \mathbb{D} \end{tikzcd}$$

LIMITS INDEXED BY LOOSE DISTRIBUTORS

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$$\begin{tikzcd}
\mathbb{I} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid"{marking}] & \mathbb{D}
\end{tikzcd}$$

is a loose morphism $\lim \Phi : \lim F \to\!\!\!\!\shortmid\ \lim G$ in $\mathbb{D}$ and a terminal cone Θ

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$$\begin{tikzcd}
\mathbb{I} \arrow[r, "P", "\shortparallel"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortparallel"{marking}] & \mathbb{D}
\end{tikzcd}$$

is a loose morphism $\lim \Phi : \lim F \relbar\joinrel\mapstochar\joinrel\rightarrow \lim G$ in $\mathbb{D}$ and a terminal cone Θ which provides for each $p : A \relbar\joinrel\mapstochar\joinrel\rightarrow X$ in $P(A, X)$

$$\begin{tikzcd}
FA \arrow[r, "\Phi p"', "\shortmid"{marking}] & GX
\end{tikzcd}$$

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\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortparallel"{marking}] & \mathbb{D}
\end{tikzcd}$$

is a loose morphism $\lim \Phi : \lim F \mathrel{\mapstochar\!\rightarrow} \lim G$ in $\mathbb{D}$ and a terminal cone θ which provides for each $p : A \mathrel{\mapstochar\!\rightarrow} X$ in $P(A,X)$

$$\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
FA \arrow[r, "\Phi p"', "\shortmid"{marking}] & GX
\end{tikzcd} \qquad \text{in } \mathbb{D}$$

natural w.r.t. heterocells of $P : \mathbb{I} \mathrel{\mapstochar\mapstochar\!\rightarrow} \mathbb{J}$ and satisfying

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\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid\shortmid"{marking}] & \mathbb{D}
\end{tikzcd}$$

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FA \arrow[r, "\Phi p"', "\shortmid"{marking}] & GX
\end{tikzcd}$$

in $\mathbb{D}$

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FA \arrow[r, "\Phi p", "\shortmid"{marking}] & GX
\end{tikzcd}$$

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\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
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FA \arrow[r, "\Phi p"', "\shortmid"{marking}] & GX
\end{tikzcd} \qquad \text{in } \mathbb{D}$$

natural w.r.t. heterocells of $P : \mathbb{I} \mathrel{\mapstochar\rightarrow} \mathbb{J}$ and satisfying

$$\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_{A'}"'] \arrow[dr, phantom, "\gamma_q"] & \lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
FA' \arrow[r, "Fq", "\shortmid"{marking}] & FA \arrow[r, "\Phi p", "\shortmid"{marking}] & GX
\end{tikzcd}
=
\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_{A'}"'] \arrow[dr, phantom, "\theta_{q \triangleright p}"] & \lim G \arrow[d, "\psi_X"] \\
FA' \arrow[r, "\Phi(q \triangleright p)"', "\shortmid"{marking}] & GX
\end{tikzcd}$$

$$\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\psi_X"] \arrow[dr, phantom, "\psi_r"] & \lim G \arrow[d, "\psi_{X'}"] \\
FA \arrow[r, "\Phi p", "\shortmid"{marking}] & GX \arrow[r, "Gr", "\shortmid"{marking}] & GX'
\end{tikzcd}
=
\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_{p \triangleleft r}"] & \lim G \arrow[d, "\psi_{X'}"] \\
FA \arrow[r, "\Phi(p \triangleleft r)"', "\shortmid"{marking}] & GX'
\end{tikzcd}$$

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Suppose $(\lim F, \gamma)$ and $(\lim G, \psi)$ are limits of $F : \mathbb{I} \longrightarrow \mathbb{D}$ and $G : \mathbb{J} \longrightarrow \mathbb{D}$, respectively. The limit of an alteration

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\end{tikzcd}$$

is a loose morphism $\lim \Phi : \lim F \nrightarrow \lim G$ in $\mathbb{D}$ and a terminal cone θ which provides for each $p : A \nrightarrow X$ in $P(A,X)$

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\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
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\end{tikzcd} \qquad \text{in } \mathbb{D}$$

natural w.r.t. heterocells of $P : \mathbb{I} \nrightarrow \mathbb{J}$ and satisfying

$$\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_{A'}"'] \arrow[dr, phantom, "\gamma_q"] & \lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
FA' \arrow[r, "Fq", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\underline{c}^{\triangleright}(q,p)"] & FA \arrow[r, "\Phi p", "\shortmid"{marking}] & GX \arrow[d, equal] \\
FA' \arrow[rr, "\Phi(q \triangleright p)"', "\shortmid"{marking}] & & GX
\end{tikzcd}
=
\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id} \odot \lim \Phi", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim G \arrow[d, equal] \\
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_{A'}"'] \arrow[dr, phantom, "\theta_{q \triangleright p}"] & \lim G \arrow[d, "\psi_X"] \\
FA' \arrow[r, "\Phi(q \triangleright p)"', "\shortmid"{marking}] & GX
\end{tikzcd}$$

$$\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\psi_X"] \arrow[dr, phantom, "\psi_r"] & \lim G \arrow[d, "\psi_{X'}"] \\
FA \arrow[r, "\Phi p", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\underline{c}^{\triangleleft}(p,r)"] & GX \arrow[r, "Gr", "\shortmid"{marking}] & GX' \arrow[d, equal] \\
FA \arrow[rr, "\Phi(p \triangleleft r)"', "\shortmid"{marking}] & & GX'
\end{tikzcd}
=
\begin{tikzcd}
\lim F \arrow[r, "\lim \Phi \odot \mathrm{id}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim G \arrow[d, equal] \\
\lim F \arrow[r, "\lim \Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_{p \triangleleft r}"] & \lim G \arrow[d, "\psi_{X'}"] \\
FA \arrow[r, "\Phi(p \triangleleft r)"', "\shortmid"{marking}] & GX'
\end{tikzcd}$$

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Suppose $(\lim F, \gamma)$ and $(\lim G, \psi)$ are limits of $F : \mathbb{I} \longrightarrow \mathbb{D}$ and $G : \mathbb{J} \longrightarrow \mathbb{D}$, respectively. The limit of an alteration

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "P", "\shortmid\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\Phi"] & \mathbb{J} \arrow[d, "G"] \\
\mathbb{D} \arrow[r, "\mathrm{Hom}"', "\shortmid\shortmid"{marking}] & \mathbb{D}
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is a loose morphism $\lim\Phi : \lim F \to \lim G$ in $\mathbb{D}$ and a terminal cone θ which provides for each $p : A \to X$ in $P(A,X)$

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FA \arrow[r, "\Phi p"', "\shortmid"{marking}] & GX
\end{tikzcd}$$

in $\mathbb{D}$

natural w.r.t. heterocells of $P : \mathbb{I} \to \mathbb{J}$ and satisfying

$$\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\gamma_{A'}"'] \arrow[dr, phantom, "\gamma_q"] & \lim F \arrow[r, "\lim\Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[d, "\psi_X"] \\
FA' \arrow[r, "Fq", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\underline{c}^{\triangleright}(q,p)"] & FA \arrow[r, "\Phi p", "\shortmid"{marking}] & GX \arrow[d, equal] \\
FA' \arrow[rr, "\Phi(q \triangleright p)"', "\shortmid"{marking}] & & GX
\end{tikzcd}
=
\begin{tikzcd}
\lim F \arrow[r, "\mathrm{id} \odot \lim\Phi", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim G \arrow[d, equal] \\
\lim F \arrow[r, "\lim\Phi", "\shortmid"{marking}] \arrow[d, "\gamma_{A'}"'] \arrow[dr, phantom, "\theta_{q \triangleright p}"] & \lim G \arrow[d, "\psi_X"] \\
FA' \arrow[r, "\Phi(q \triangleright p)"', "\shortmid"{marking}] & GX
\end{tikzcd}$$

$$\begin{tikzcd}
\lim F \arrow[r, "\lim\Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_p"] & \lim G \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\psi_X"] \arrow[dr, phantom, "\psi_r"] & \lim G \arrow[d, "\psi_{X'}"] \\
FA \arrow[r, "\Phi p", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\underline{c}^{\triangleleft}(p,r)"] & GX \arrow[r, "Gr", "\shortmid"{marking}] & GX' \arrow[d, equal] \\
FA \arrow[rr, "\Phi(p \triangleleft r)"', "\shortmid"{marking}] & & GX'
\end{tikzcd}
=
\begin{tikzcd}
\lim F \arrow[r, "\lim\Phi \odot \mathrm{id}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & \lim G \arrow[d, equal] \\
\lim F \arrow[r, "\lim\Phi", "\shortmid"{marking}] \arrow[d, "\gamma_A"'] \arrow[dr, phantom, "\theta_{p \triangleleft r}"] & \lim G \arrow[d, "\psi_{X'}"] \\
FA \arrow[r, "\Phi(p \triangleleft r)"', "\shortmid"{marking}] & GX'
\end{tikzcd}$$

⚠ Limits of alterations can be pathological unless $\mathbb{D}$ is replete : $\langle \mathrm{dom}, \mathrm{cod} \rangle : D_1 \longrightarrow D_0 \times D_0$ is an isofibration.

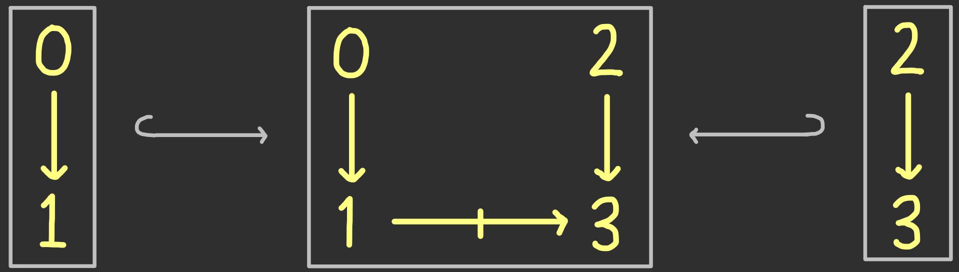
COMPANIONS, CONJOINTS, & RESTRICTIONS ARE LIMITS

- A restriction is a limit whose shape is (the collage)

$$\begin{tikzcd}
0 \arrow[dd] & & 0 \arrow[dd] & & 2 \arrow[dd] & & 2 \arrow[dd] \\
& {} \arrow[r, hook] & {} & & {} & {} \arrow[l, hook'] & \\
1 & & 1 \arrow[rr, "|" marking] & & 3 & & 3
\end{tikzcd}$$

COMPANIONS, CONJOINTS, & RESTRICTIONS ARE LIMITS

- A restriction is a limit whose shape is (the collage)



$$\begin{tikzcd} A \arrow[r, "\mathrm{lim}", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{res}"] & C \arrow[d, "g"] \\ B \arrow[r, "h"', "|" marking] & D \end{tikzcd}$$

We choose the limit of a tight morphism to be its domain.

COMPANIONS, CONJOINTS, & RESTRICTIONS ARE LIMITS

- A restriction is a limit whose shape is (the collage)

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & & 2 \arrow[d] & & 2 \arrow[d] \\
1 \arrow[rr, hook] & & 1 \arrow[rr, "|" marking] & & 3 & & 3 \arrow[ll, hook']
\end{tikzcd}$$

- The universal property states (assuming repleteness)

$$\begin{tikzcd}
X \arrow[r, "q", "|" marking] \arrow[d, "h"'] & Y \arrow[d, "k"] & & & \\
A \arrow[d, "f"'] \arrow[r, phantom, "\forall \alpha"] & C \arrow[d, "g"] & & A \arrow[r, "\mathrm{lim}", "|" marking] \arrow[d, "f"'] \arrow[r, phantom, "\mathrm{res}", shift right=6ex] & C \arrow[d, "g"] \\
B \arrow[r, "h"', "|" marking] & D & & B \arrow[r, "h"', "|" marking] & D
\end{tikzcd}$$

We choose the limit of a tight morphism to be its domain.

COMPANIONS, CONJOINTS, & RESTRICTIONS ARE LIMITS

- A restriction is a limit whose shape is (the collage)

$$\begin{tikzcd}
\boxed{\begin{matrix} 0 \\ \downarrow \\ 1 \end{matrix}} \arrow[r, hook] & \boxed{\begin{tikzcd} 0 \arrow[d] & 2 \arrow[d] \\ 1 \arrow[r, "|" marking] & 3 \end{tikzcd}} & \boxed{\begin{matrix} 2 \\ \downarrow \\ 3 \end{matrix}} \arrow[l, hook']
\end{tikzcd}$$

- The universal property states (assuming repleteness)

$$\begin{tikzcd}
X \arrow[r, "q", "|" marking] \arrow[d, "h"'] & Y \arrow[d, "k"] \\
A \arrow[d, "f"'] \arrow[r, phantom, "\forall \alpha"] & C \arrow[d, "g"] \\
B \arrow[r, "h"', "|" marking] & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "q", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\exists ! \hat{\alpha}"] & Y \arrow[d, "k"] \\
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We choose the limit of a tight morphism to be its domain.

COMPANIONS, CONJOINTS, & RESTRICTIONS ARE LIMITS

- A restriction is a limit whose shape is (the collage)

$$\begin{tikzcd}
\boxed{\begin{array}{c} 0 \\ \downarrow \\ 1 \end{array}} \arrow[r, hook] & \boxed{\begin{array}{ccc} 0 & & 2 \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\shortmid} & 3 \end{array}} & \boxed{\begin{array}{c} 2 \\ \downarrow \\ 3 \end{array}} \arrow[l, hook']
\end{tikzcd}$$

- The universal property states (assuming repleteness)

$$\begin{tikzcd}
X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, "h"'] & Y \arrow[d, "k"] & & X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, "h"'] & Y \arrow[d, "k"] \\
A \arrow[d, "f"'] \arrow[r, phantom, "\forall \alpha"] & C \arrow[d, "g"] & = & A \arrow[r, "\mathrm{lim}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[ur, phantom, "\exists ! \hat{\alpha}"] & C \arrow[d, "g"] \\
B \arrow[r, "h"', "\shortmid"{marking}] & D & & B \arrow[r, "h"', "\shortmid"{marking}] \arrow[ur, phantom, "\mathrm{res}"] & D
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- A double category admits companions if and only if it admits limits whose shape is

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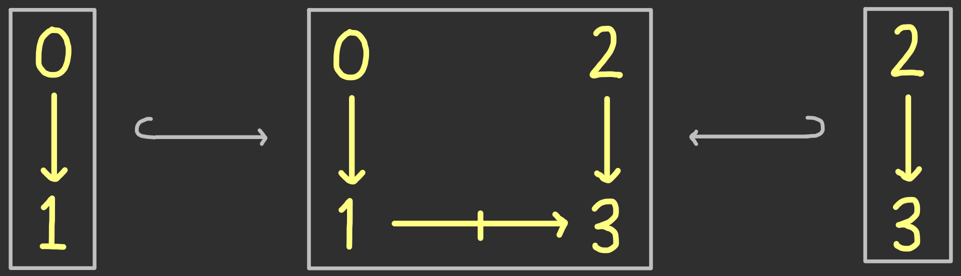

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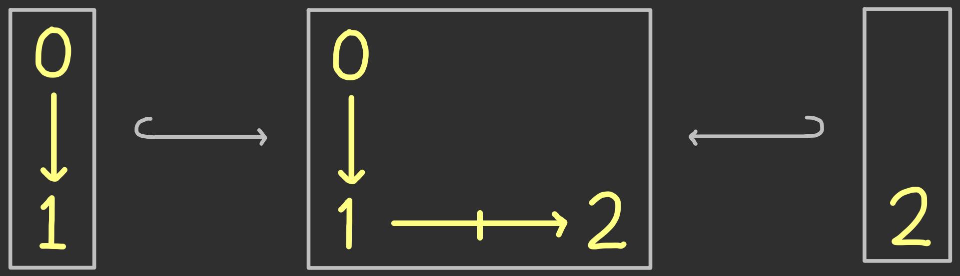

- A double category admits restrictions if and only if it admits companions and conjoints.

COMPANIONS, CONJOINTS, & RESTRICTIONS ARE LIMITS

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$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & & 2 \arrow[d] & & 2 \arrow[d] \\
1 & \hookrightarrow & 1 \arrow[rr, "\shortmid" marking] & & 3 & \hookleftarrow & 3
\end{tikzcd}$$

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$$\begin{tikzcd}
X \arrow[r, "q", "\shortmid" marking] \arrow[d, "h"'] & Y \arrow[d, "k"] & & X \arrow[r, "q", "\shortmid" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\exists!\hat{\alpha}"] & Y \arrow[d, "k"] \\
A \arrow[r, phantom, "\forall\alpha"] \arrow[d, "f"'] & C \arrow[d, "g"] & = & A \arrow[r, "\text{lim}", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\text{res}"] & C \arrow[d, "g"] \\
B \arrow[r, "h"', "\shortmid" marking] & D & & B \arrow[r, "h"', "\shortmid" marking] & D
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- ℝel, $pan, and 𝔻ist admit all restrictions. E.g. in ℝel

$$\begin{tikzcd}
A \times C \arrow[d, "f \times g"'] \arrow[dr, dashed, bend left, "R(f-{,}g-)"] & \\
B \times D \arrow[r, "R"'] & \{\perp \to \top\}
\end{tikzcd}$$

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- $\mathbb{R}\mathrm{el}$, $\mathbb{S}\mathrm{pan}$, and $\mathbb{D}\mathrm{ist}$ admit all restrictions. E.g. in $\mathbb{R}\mathrm{el}$

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- Restrictions, etc. are preserved by any unitary lax functor.

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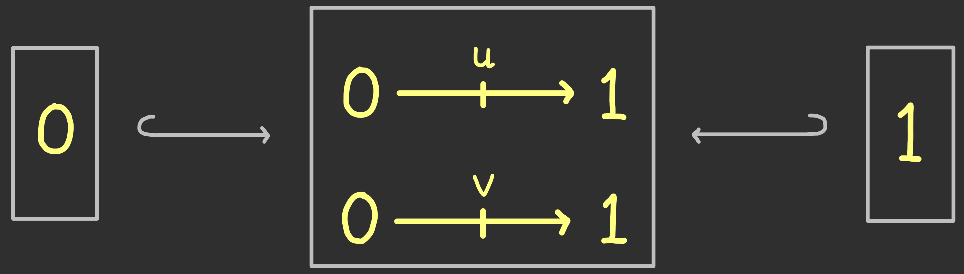

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- The local product of $p_1, p_2 : A \nrightarrow B$ is a loose morphism $p_1 \wedge p_2 : A \nrightarrow B$ and projection cells

$$\boxed{i=1,2} \qquad \begin{tikzcd}
A \arrow[r, "p_1 \wedge p_2"] \arrow[d, equal] \arrow[dr, phantom, "\pi_i"] & B \arrow[d, equal] \\
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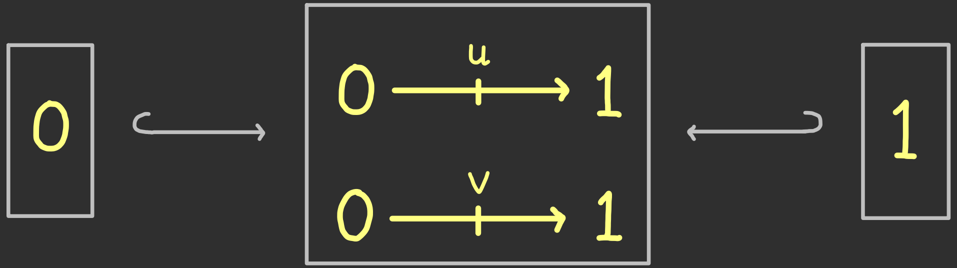

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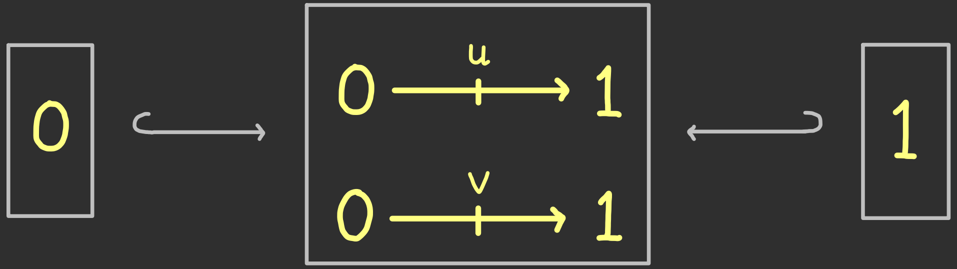

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$$\begin{tikzcd} A \arrow[r, "p_1 \wedge p_2", "\shortmid" marking] \arrow[d, equal] & B \arrow[r, "q", "\shortmid" marking] \arrow[d, phantom, "\exists!"] & C \arrow[d, equal] \\ A \arrow[rr, "(p_1 \odot q) \wedge (p_2 \odot q)"', "\shortmid" marking] & & C \end{tikzcd}$$

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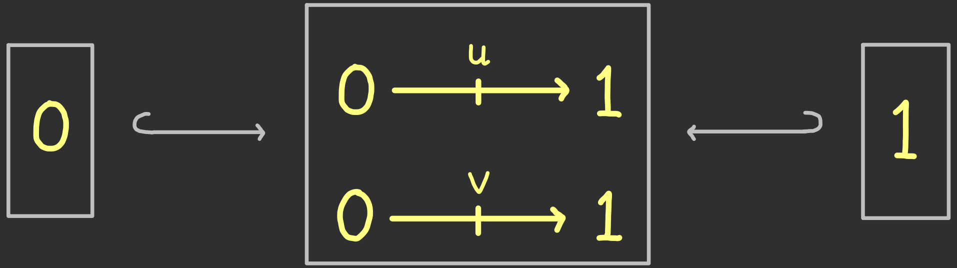

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$$\begin{tikzcd}
A \times B \arrow[r, "\langle R, S \rangle"] \arrow[dr, dashed, "R \wedge S"'] & \{\bot \to \top\} \times \{\bot \to \top\} \arrow[d, "\wedge"] \\
& \{\bot \to \top\}
\end{tikzcd}$$

local product in $\mathbb{R}\mathrm{el}$

HOMOLOGOUS & TIGHT PARALLEL LIMITS

- A homologous limit is a limit whose shape is

$$\Pi_i(I) \longrightarrow\!\!\!\!\!\!\shortparallel\!\!\!\!\longrightarrow \Pi_i(J) \qquad I, J \text{ categories}$$

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$$\begin{tikzcd}
I \arrow[d, "F"'] & P \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\Phi"] & J \arrow[d, "G"] \\
D_0 & D_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & D_0
\end{tikzcd} \quad \text{in } \mathrm{CAT}$$

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- A double category has tight parallel limits if and only if it admits parallel products & parallel equalisers.

HOMOLOGOUS & TIGHT PARALLEL LIMITS

- A **homologous limit** is a limit whose shape is

$$\begin{tikzcd} \mathbb{T}\mathrm{i}(I) \arrow[r, "|" marking] & \mathbb{T}\mathrm{i}(J) \end{tikzcd} \qquad I, J \text{ categories}$$

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$$\begin{tikzcd} \mathbb{T}\mathrm{i}(I) \arrow[r, "\mathrm{Hom}", "|" marking] & \mathbb{T}\mathrm{i}(I) \end{tikzcd} \qquad I \text{ category}$$

- A double category has tight parallel limits if and only if it admits parallel products & parallel equalisers.

- E.g. the parallel product of $p : A \mathrel{-\mkern-6mu\mapstochar\mkern-6mu\rightarrow} B$ and $q : C \mathrel{-\mkern-6mu\mapstochar\mkern-6mu\rightarrow} D$ is

$$\begin{tikzcd} A \times C \arrow[r, "p \times q", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & B \times D \arrow[d, "\pi_B"] \\ A \arrow[r, "p"', "|" marking] & B \end{tikzcd} \qquad \begin{tikzcd} A \times C \arrow[r, "p \times q", "|" marking] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\pi_q"] & B \times D \arrow[d, "\pi_D"] \\ C \arrow[r, "q"', "|" marking] & D \end{tikzcd}$$

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$$\begin{tikzcd} \mathbb{T}\mathrm{i}(I) \arrow[r, "\shortmid\shortmid" marking] & \mathbb{T}\mathrm{i}(J) \end{tikzcd} \qquad I, J \text{ categories}$$

- An alteration of this shape into $\mathbb{D}$ is precisely

$$\begin{tikzcd} I & P \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\Phi"] & J \arrow[d, "G"] \\ D_0 \arrow[from=u, "F"'] & D_1 \arrow[l, "dom"] \arrow[r, "cod"'] & D_0 \end{tikzcd} \quad \text{in CAT}$$

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Theorem: A double category admits homologous limits if and only if it admits tight parallel limits and restrictions.

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$\prod_i(I) \nrightarrow \prod_i(J)$ is a span

- Restrictions and local limits are examples.

- A tight parallel limit is a limit whose shape is

$$\begin{tikzcd} \prod_i(I) \arrow[r, "Hom", "\|" marking] & \prod_i(I) \end{tikzcd} \qquad I \text{ category}$$

- A double category has tight parallel limits if and only if it admits parallel products & parallel equalisers.

- E.g. the parallel product of $p : A \nrightarrow B$ and $q : C \nrightarrow D$ is

$$\begin{tikzcd} A \times C \arrow[r, "p \times q", "|" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & B \times D \arrow[d, "\pi_B"] \\ A \arrow[r, "|" marking, "p"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \times C \arrow[r, "p \times q", "|" marking] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\pi_q"] & B \times D \arrow[d, "\pi_D"] \\ C \arrow[r, "|" marking, "q"'] & D \end{tikzcd}$$

Theorem: A double category admits homologous limits if and only if it admits tight parallel limits and restrictions.

Corollary: $\mathbb{D}$ has homologous limits if and only if D_0 and D_1 have limits preserved by $dom, cod : D_1 \rightrightarrows D_0$ and $\langle dom, cod \rangle : D_1 \longrightarrow D_0 \times D_0$ is a fibration.

HOMOLOGOUS & TIGHT PARALLEL LIMITS

- A homologous limit is a limit whose shape is

$$\begin{tikzcd} \prod_i(I) \arrow[r, "\shortmid\shortmid" marking] & \prod_i(J) \end{tikzcd} \qquad I, J \text{ categories}$$

- An alteration of this shape into $\mathbb{D}$ is precisely

$$\begin{tikzcd} I \arrow[d, "F"'] & P \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\Phi"] & J \arrow[d, "G"] \\ D_0 & D_1 \arrow[l, "\mathrm{dom}"] \arrow[r, "\mathrm{cod}"'] & D_0 \end{tikzcd} \qquad \text{in CAT}$$

$\prod_i(I) \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} \prod_i(J)$ is a span

- Restrictions and local limits are examples.

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$$\begin{tikzcd} A \times C \arrow[r, "p \times q", "\shortmid" marking] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_p"] & B \times D \arrow[d, "\pi_B"] & & A \times C \arrow[r, "p \times q", "\shortmid" marking] \arrow[d, "\pi_C"'] \arrow[dr, phantom, "\pi_q"] & B \times D \arrow[d, "\pi_D"] \\ A \arrow[r, "p"', "\shortmid" marking] & B & & C \arrow[r, "q"', "\shortmid" marking] & D \end{tikzcd}$$

Theorem: A double category admits homologous limits if and only if it admits tight parallel limits and restrictions.

Corollary: $\mathbb{D}$ has homologous limits if and only if D_0 and D_1 have limits preserved by $\mathrm{dom}, \mathrm{cod} : D_1 \rightrightarrows D_0$ and $\langle \mathrm{dom}, \mathrm{cod} \rangle : D_1 \longrightarrow D_0 \times D_0$ is a fibration.

Corollary: A double category admits local XX if it admits parallel XX and restrictions.
XX = products, equalisers, etc.

PARALLEL TABULATORS & MAIN THEOREM

- A parallel tabulator is a limit whose shape is

$$\begin{tikzcd}\mathbb{2} \arrow[r, "\mathrm{Hom}", "\shortparallel" marking] & \mathbb{2}\end{tikzcd}$$

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- An alteration with this shape determines cells in $\mathbb{D}$

$$\begin{tikzcd} A \arrow[r, "p", "|" marking] \arrow[dd, equal] & B \arrow[r, "r_2", "|" marking] & D \arrow[dd, equal] \\ & \alpha & \\ A \arrow[rr, "r_3"', "|" marking] & & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "r_1", "|" marking] \arrow[dd, equal] & C \arrow[r, "q", "|" marking] & D \arrow[dd, equal] \\ & \beta & \\ A \arrow[rr, "r_3"', "|" marking] & & D \end{tikzcd}$$

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whose parallel tabulator is a loose morphism $Tp \xrightarrow{\shortmid} Tq$ between tabulators and a cone given by cells

$$\begin{tikzcd} Tp \arrow[r, "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_{r_1}"] & Tq \arrow[d, "\pi_C"] \\ A \arrow[r, "r_1"', "\shortmid"{marking}] & C \end{tikzcd} \qquad \begin{tikzcd} Tp \arrow[r, "\shortmid"{marking}] \arrow[d, "\pi_B"'] \arrow[dr, phantom, "\pi_{r_2}"] & Tq \arrow[d, "\pi_D"] \\ B \arrow[r, "r_2"', "\shortmid"{marking}] & D \end{tikzcd} \qquad \begin{tikzcd} Tp \arrow[r, "\shortmid"{marking}] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\pi_{r_3}"] & Tq \arrow[d, "\pi_D"] \\ A \arrow[r, "r_3"', "\shortmid"{marking}] & D \end{tikzcd}$$

which are suitably compatible with α and β.

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Theorem (Grandis-Paré, 99)

A double category admits parallel limits if and only if it admits **parallel tabulators** and **tight parallel limits**.

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Theorem (Grandis-Paré, 99)
A double category admits parallel limits if and only if it admits parallel tabulators and tight parallel limits.

Theorem: A double category $\mathbb{D}$ admits limits indexed by loose distributors if and only if

(1) $\mathbb{D}$ admits parallel limits and restrictions

if and only if

(2) $\mathbb{D}$ admits parallel tabulators and homologous limits.

SUMMARY & FURTHER WORK

- Introduced a new framework for limits in double categories indexed by loose distributors $\mathbb{I} \xrightarrow{P} \mathbb{J}$.

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- Introduced a new framework for limits in double categories indexed by loose distributors $\mathbb{I} \overset{P}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbb{J}$.

- Captures many well-known concepts as examples:
  - * Parallel limits $\mathbb{I} \overset{\mathrm{Hom}}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbb{I}$ and parallel tabulators $\mathbb{2} \overset{\mathrm{Hom}}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbb{2}$.
  - * Restrictions $\mathbb{T}\mathrm{i}(2) \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{T}\mathrm{i}(2)$, companions and conjoints.
  - * Local limits $\mathbb{1} \overset{P}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbb{1}$, including local products
  - * Homologous limits $\mathbb{T}\mathrm{i}(\mathbb{C}) \overset{P}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbb{T}\mathrm{i}(\mathbb{C})$.

SUMMARY & FURTHER WORK

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  - * Local limits $\mathbb{1} \overset{P}{\nrightarrow} \mathbb{1}$, including local products
  - * Homologous limits $\mathbb{T}_i(C) \overset{P}{\nrightarrow} \mathbb{T}_i(C)$.

Theorem: A double category $\mathbb{D}$ admits limits indexed by loose distributors if and only if

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Theorem: A double category $\mathbb{D}$ admits limits indexed by loose distributors if and only if

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Many current and future research directions.

- Sufficient conditions for completeness of $\mathbb{S}\mathrm{pan}(\mathcal{E})$, $\mathbb{R}\mathrm{el}(\mathcal{E})$, $\mathbb{M}\mathrm{at}(\mathbb{D})$, $\mathbb{M}\mathrm{od}(\mathbb{D})$, etc.
- Interactions between limits indexed by double categories and limits indexed by loose distributors.
- Relationship with bicategorical (co)limits.
- Constructing (co)completions of double categories.
- Extending Lambert-Patterson's Cartesian double theories to a general framework of double-categorical sketches.
- Characterising the class of absolute (co)limits.
- Generalisation to virtual double categories.

BONUS SLIDE: LAX BICATEGORICAL COLIMITS

- Each $\mathbb{I} \xrightarrow{!} \mathbb{1}$ determines canonical loose distributors:

$$\begin{tikzcd} \mathbb{I} \arrow[r, "\top", "\shortmid" marking] & \mathbb{1} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "\top", "\shortmid" marking] & \mathbb{I} \end{tikzcd}$$

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- Example: consider the colimit of $\{1\ 2\} \xrightarrow{\top} \mathbb{1}$ whose diagram is a pair of loose morphisms $A_1 \xrightarrow{p_1} X \xleftarrow{p_2} A_2$

$$\begin{tikzcd} A_i \arrow[r, "p_i", "\shortmid"{marking}] \arrow[d, "\amalg_i"'] \arrow[dr, phantom, "\theta_i"] & X \arrow[d, equal] \\ A_1 + A_2 \arrow[r, "\operatorname{colim}(p)"', "\shortmid"{marking}] & X \end{tikzcd}$$

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and the colimit of $\mathbb{1} \overset{\top}{\nrightarrow} \{1\ 2\}$ given by:

$$\begin{tikzcd} Y \arrow[r, "q_i", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\omega_i"] & A_i \arrow[d, "\amalg_i"] \\ Y \arrow[r, "\mathrm{colim}(q)"', "|"{marking}] & A_1+A_2 \end{tikzcd}$$

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- If $\mathbb{D}$ has companions and conjoints we obtain cells:

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- If $\mathbb{D}$ has companions and conjoints we obtain cells:

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$$\begin{tikzcd} A \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, equal] & A_i \arrow[r, "p_i", "\shortmid"{marking}] \arrow[d, "\amalg_i"'] \arrow[dr, phantom, "\theta_i"] & X \arrow[d, equal] \\ A \arrow[r, "(\amalg_i)_*"', "\shortmid"{marking}] & A_1 + A_2 \arrow[r, "\mathrm{colim}(p)"', "\shortmid"{marking}] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "q_i", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\omega_i"] & A_i \arrow[r, "\mathrm{id}", "\shortmid"{marking}] \arrow[d, "\amalg_i"] & A_i \arrow[d, equal] \\ Y \arrow[r, "\mathrm{colim}(q)"', "\shortmid"{marking}] & A_1 + A_2 \arrow[r, "(\amalg_i)^*"', "\shortmid"{marking}] & A_i \end{tikzcd}$$

BONUS SLIDE: LAX BICATEGORICAL COLIMITS

- Each $\mathbb{I} \xrightarrow{!} \mathbb{1}$ determines canonical loose distributors:

$$\begin{tikzcd} \mathbb{I} \arrow[r, "\top", "\|" marking] & \mathbb{1} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "\top", "\|" marking] & \mathbb{I} \end{tikzcd}$$

- Example: consider the colimit of $\{1\ 2\} \xrightarrow{\top} \mathbb{1}$ whose diagram is a pair of loose morphisms $A_1 \xrightarrow{p_1} X \xleftarrow{p_2} A_2$

$$\begin{tikzcd} A_i \arrow[r, "p_i", "|" marking] \arrow[d, "\amalg_i"'] \arrow[dr, phantom, "\theta_i"] & X \arrow[d, equal] \\ A_1 + A_2 \arrow[r, "\mathrm{colim}(p)"', "|" marking] & X \end{tikzcd}$$

and the colimit of $\mathbb{1} \xrightarrow{\top} \{1\ 2\}$ given by:

$$\begin{tikzcd} Y \arrow[r, "q_i", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\omega_i"] & A_i \arrow[d, "\amalg_i"] \\ Y \arrow[r, "\mathrm{colim}(q)"', "|" marking] & A_1 + A_2 \end{tikzcd}$$

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- In $\mathbb{R}\mathrm{el}$, $\mathbb{S}\mathrm{pan}$, and $\mathbb{D}\mathrm{ist}$ these cells are invertible, and describe the coproduct and product in the underlying bicategory of $\mathbb{D}$ — which coincide!

BONUS SLIDE: LAX BICATEGORICAL COLIMITS

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- Takeaway: biproducts in Rel are colimits in $\mathbb{R}$el.

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$$\begin{tikzcd} A_i \arrow[r, "p_i", "\shortmid" marking] \arrow[d, "\perp\!\!\!\perp_i"'] \arrow[dr, phantom, "\theta_i"] & X \arrow[d, equal] \\ A_1+A_2 \arrow[r, "\mathrm{colim}(p)"', "\shortmid" marking] & X \end{tikzcd}$$

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- In $\mathbb{R}\mathrm{el}$, $\mathbb{S}\mathrm{pan}$, and $\mathbb{D}\mathrm{ist}$ these cells are invertible, and describe the coproduct and product in the underlying bicategory of $\mathbb{D}$ — which coincide!

- Takeaway: biproducts in Rel are colimits in $\mathbb{R}\mathrm{el}$.

Conjecture: Let $\mathbb{D}$ have companions & conjoints. A (unitary colax) functor $\mathbb{J}$ admits a lax colimit if and only if alteration from $\mathbb{J} \xrightarrow{\perp} \mathbb{1}$ admits a colimit.