On lax comma categories

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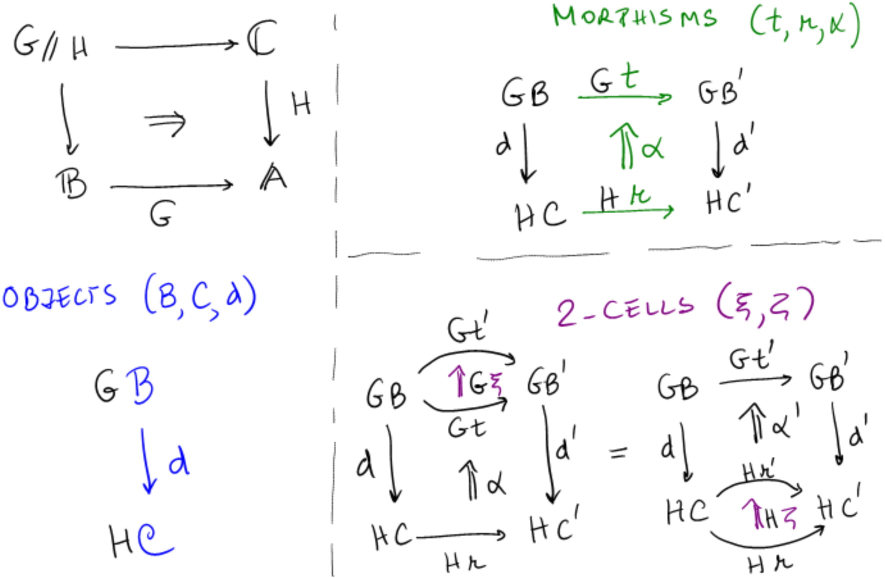
Centre for Mathematics of the University of Coimbra

17th July 2025

(Joint work with Fernando Lucatelli Nunes, George Janelidze, Rui Prezado, Dirk Hofmann.)

Why?

The lax comma category $G//H$



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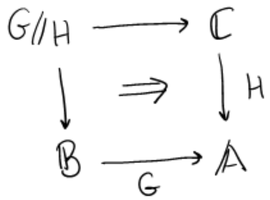

EXAMPLES

① $\mathbb{A} // X$

$\mathbb{C} = 1,\ H(*) = X,\ G = \mathrm{Id}_{\mathbb{A}}$

Eg: V-normed categories: $\mathrm{Cat} // V$

② $X // \mathbb{A}$

③ Grothendieck construction $\int A$

$\mathbb{A} = \mathrm{CAT},\ \mathbb{B} = 1,\ H = A$

④ Lax scones

Plan

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Cartesian closedness, extensivity, topologicity, and descent...

- Cat//$\boldsymbol{X}$

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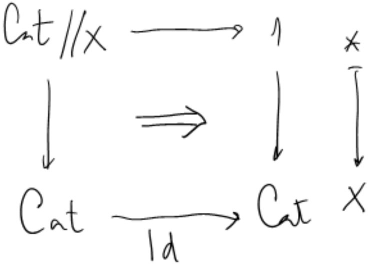
- $\boldsymbol{G}/\!/\boldsymbol{H}$.

Cat//X



Cat//X: (co)limits

C-Lucatelli-Prezado, Lax comma categories: cartesian closedness, extensivity, topologicity, and descent, TAC 2024

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If $\boldsymbol{X}$ has initial object, then $\mathrm{Cat}/\!/\boldsymbol{X}$ has coproducts (preserved by $\boldsymbol{U}$).

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If X has initial object, then $\mathrm{Cat}/\!/X$ has coproducts (preserved by U).

If X has initial object, then $\mathrm{Cat}/\!/X$ is (infinitary) extensive.

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Cat//X: exponentiability and descent

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Thm. If X is complete and cartesian closed, then Cat//X is cartesian closed.

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The exponential $(W, a) \Rightarrow (Y, b)$ is given by $(\mathrm{Cat}[W, Y], b^a)$, with

$$b^a(h) = \int_{w \in W} (a(w) \Rightarrow b \cdot h(w)).$$

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Recall:

In a category A with pullbacks, a morphism $y \xrightarrow{f} z$ is effective for descent

if the change-of-base functor $A/z \xrightarrow{f^*} A/y$ is monadic.

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Thm. If X has pullbacks, the functor $\mathrm{Cat}//X \xrightarrow{U} \mathrm{Cat}$ preserves effective descent

morphisms provided that X has a strict initial object.

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Ord//X

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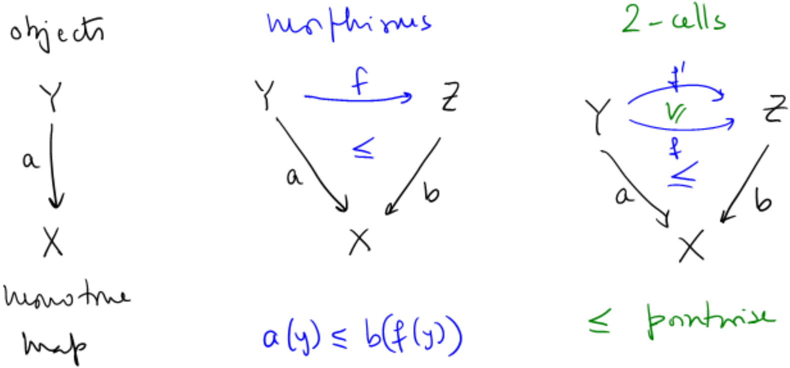
Ord//X



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$\mathrm{Ord}/\!/\boldsymbol{X}$: (co)limits

Prop. The forgetful functor $\mathrm{Ord}/\!/\boldsymbol{X} \xrightarrow{U} \mathrm{Ord}$

- is a fibration;
- has a left adjoint if $\boldsymbol{X}$ has a bottom element;
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Cor.

- If X has bottom, X is complete iff Ord//X is complete.
- If X has top, then X is complete iff Ord//X is cocomplete.

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(Remark: Coequalisers are built in Ord, and then equipped with the left Kan extension.)

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Prop. If $\boldsymbol{X}$ has bottom element, then $\mathrm{Ord}/\!/\boldsymbol{X}$ is (infinitary) extensive.

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In fact, the following conditions are equivalent, for (Y, a) in Ord$/\!/X$:

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$\mathrm{Ord}/\!/X$: exponentiability

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In fact, the following conditions are equivalent, for (Y, a) in $\mathrm{Ord}/\!/X$:

- (Y, a) is exponentiable in $\mathrm{Ord}/\!/X$;
- for all $y \in Y$, $a(y)$ is exponentiable in X.

Ord: (effective) descent monotone maps

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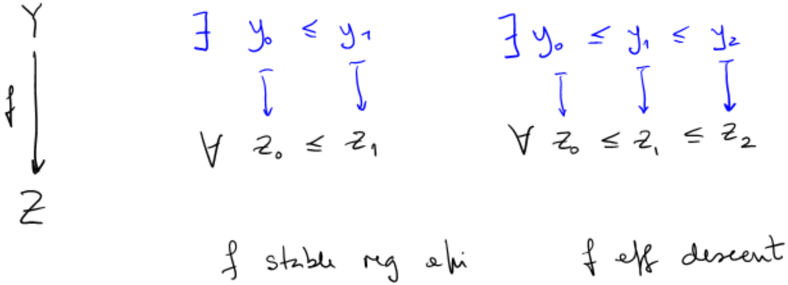

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Ord//X: (stable) regular epimorphisms

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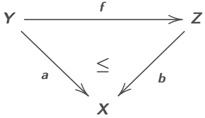
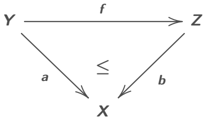
Ord//X: (stable) regular epimorphisms

Prop. Let X be complete, $f\colon (Y,a) \to (Z,b)$ a morphism in Ord//X.



Prop. Let $\boldsymbol{X}$ be complete, $\boldsymbol{f} : (\boldsymbol{Y}, \boldsymbol{a}) \to (\boldsymbol{Z}, \boldsymbol{b})$ a morphism in Ord$/\!/\boldsymbol{X}$.

(1) $\boldsymbol{f}$ is a regular epimorphism in Ord$/\!/\boldsymbol{X}$ iff:



Prop. Let X be complete, $f\colon (Y,a) \to (Z,b)$ a morphism in Ord//X.

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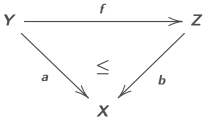

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(1) f is a regular epimorphism in Ord//X iff:

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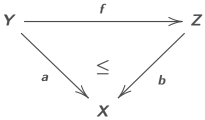

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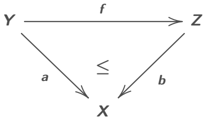

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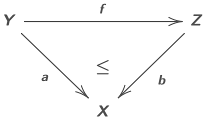

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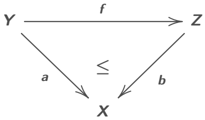

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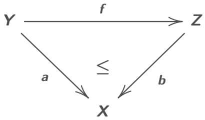
Ord//X: (stable) regular epimorphisms

Prop. Let X be locally complete with $\bot$, $f\colon (Y, a) \to (Z, b)$ a morphism in Ord//X.



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C-Prezado, Effective descent morphisms of ordered families, QM 2025

$\mathrm{Ord}/\!/X$: (stable) regular epimorphisms

Prop. Let X be locally complete with $\perp$, $f\colon (Y,a) \to (Z,b)$ a morphism in $\mathrm{Ord}/\!/X$.

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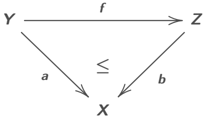

(2) f is a stable regular epimorphism in $\mathrm{Ord}/\!/X$ iff:

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Ord//X: effective descent morphisms

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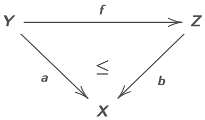

X complete

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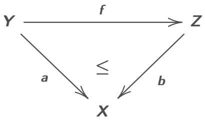
$\mathsf{Ord}//X$: effective descent morphisms



X complete

$$\boxed{f \text{ effective for descent in } \mathsf{Ord}//X}$$
$$\Downarrow$$
$$\boxed{f \text{ effective for descent in } \mathsf{Ord}}$$

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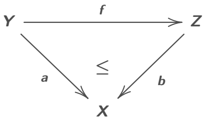

X complete

$Y_x = f^{-1}(\uparrow x) = \{y \in Y \mid x \leq a(y)\}$

f effective for descent in Ord//X

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f effective for descent in Ord

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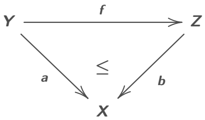
Ord$//X$: effective descent morphisms



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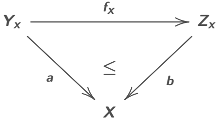

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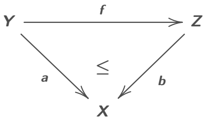
Ord//X: effective descent morphisms



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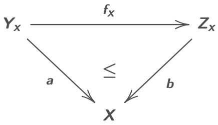

f_x effective for descent in Ord $(\forall x)$

$$\Downarrow$$

f effective for descent in Ord//X

$$\Downarrow$$

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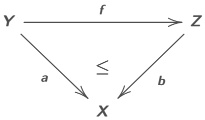

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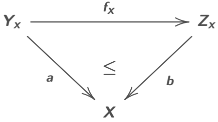

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f effective for descent in Ord//X

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$\mathrm{Ord}//X \longrightarrow [X^{op}, \mathrm{Ord}]$

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Ord//$\boldsymbol{X}$: effective descent morphims

C-Janelidze, Effective descent morphisms of filtered preorders, Order 2025
C-Lucatelli, Lax comma categories of ordered sets, QM 2023

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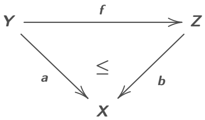

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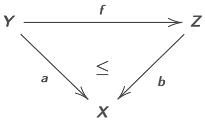

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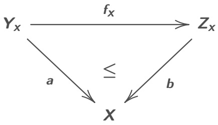

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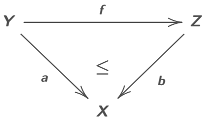

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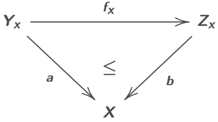

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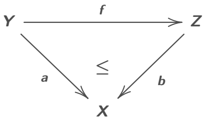

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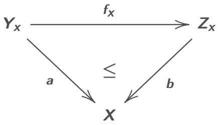

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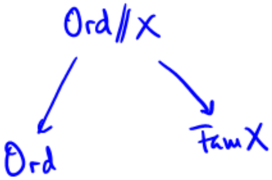

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Cor. Let $\boldsymbol{X}$ be locally complete with $\bot$, and locally cartesian closed.

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Top//X

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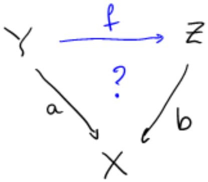

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Top//X: examples

Special examples:

- If X is a complete ordered set equipped with the lower topology, then it is a topological $\bigwedge$-semilattice.
- If X is an injective space, then it is a topological $\bigwedge$-semilattice; moreover, the map $\vee \colon X \times X \to X$ is continuous.

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  Remark: iff the product $(X, 1_X) \times (X, 1_X)$ exists in Top//X and has $X \times X$ as underlying space.
- Top//X has and U preserves limits iff X is a topological $\bigwedge$-semilattice.
- Top//X has coequalisers and U preserves them iff

  for every $Y \xrightarrow{a} X$ continuous and $Y \xrightarrow{q} Q$ quotient, $\mathrm{Lan}_q(a)$ exists.

  Remark: This is the case when X is a topological $\bigwedge$-semilattice.

Top//X: exponentiability

Prop. If X is a topological $\wedge$-semilattice with bottom, then:

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Top: effective descent morphisms

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Reiterman-Tholen, Effective descent maps of topological spaces,, TA 1994

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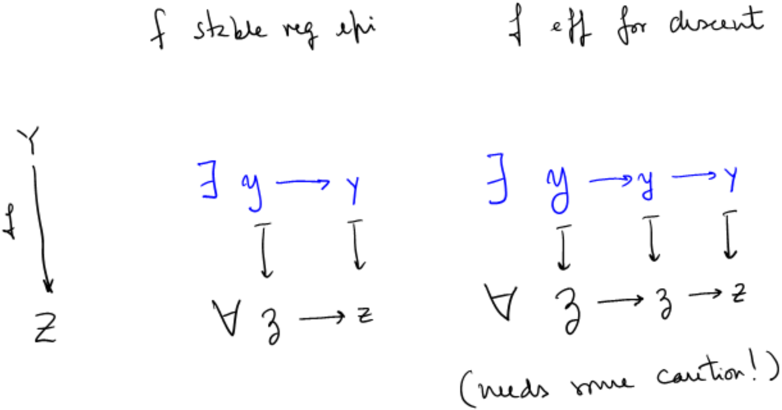

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topological

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Example: The result applies when X is a completely distributive lattice equipped with the lower topology.

The general case: $G//H$

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$G//H$: products

$G/\!\!/H$: products

Thm. Let $G\colon \mathbb{B} \to \mathbb{A}$ and $H\colon \mathbb{C} \to \mathbb{A}$ be 2-functors,

and $(B_i, C_i,\ GB_i \xrightarrow{d_i} HC_i\ )_{i\in I}$ a family of objects of $G/\!\!/H$ such that:

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Then the product of $(B_i, C_i, d_i)_i$ exists in $G/\!/B$, and it is given by

$$\Big(\prod_{i\in I} B_i, \prod_{i\in I} C_i,\ G(\textstyle\prod_{i\in I} B_i) \xrightarrow{\langle d_i\ Gp_i\rangle} \prod_{i\in I} HC_i \xrightarrow{r} H(\prod_{i\in I} C_i\ )\Big)$$

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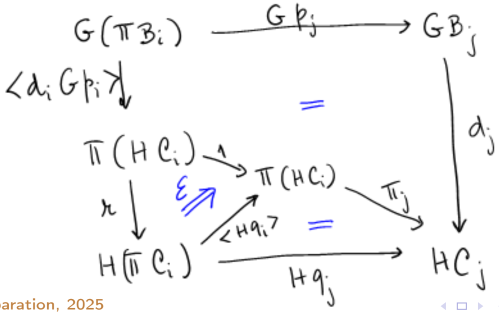

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Applying this result to the category $H^{\mathrm{co\,op}} /\!/ G^{\mathrm{co\,op}}$ one obtains:

$G /\!/ H$: coproducts

Thm. Let $G\colon \mathbb{B} \to \mathbb{A}$ and $H\colon \mathbb{C} \to \mathbb{A}$ be 2-functors,

and $(B_i, C_i,\ GB_i \xrightarrow{d_i} HC_i\ )_{i\in I}$ a family of objects of $G /\!/ H$ such that:

- $\coprod_{i\in I} B_i$ exists in $\mathbb{B}$;
- $\coprod_{i\in I} C_i$ exists in $\mathbb{C}$, with coprojections $c_j\colon C_j \to \coprod_{i\in I} C_i$;
- $\coprod_{i\in I} HC_i$ exists in $\mathbb{A}$,
- $\langle Hc_i\rangle_i\colon\ \coprod_{i\in I} HC_i \to H(\coprod_{i\in i} C_i)$ has a left adjoint s in $\mathbb{A}$.

Then the coproduct of $(B_i, C_i, d_i)_i$ exists in $G /\!/ H$.

C-Lucatelli, in preparation, 2025

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Thank you!