Projective crossed modules in semi-abelian categories

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Motivation

At the last CT in Spain, I presented my main result:

Theorem [CRVdL25]

Let $\mathcal{C}$ be a semi-abelian category and enough projectives that satisfy **Condition** (P). Let $\mathcal{E}$ be a semi-abelian category,
and let $F\colon \mathcal{C} \to \mathcal{E}$ be a protoadditive functor that preserves binary coproducts and proper morphisms.
Then the left-derived functors of F are defined as in the abelian context.

- protoadditive = a functor preserving kernels of split epimorphisms [EG10, EG15].

Example

$\pi_0\colon \mathrm{XMod}(\mathcal{V}) \to \mathcal{V}$ where $\mathcal{V}$ is a semi-abelian variety **satisfying the Condition** (P).

Condition (P)

All the categories are assumed to be **semi-abelian** throughout my talk.

Definition [CRVdL25]

We call (P) the statement that for each split short exact sequence

$$0 \longrightarrow K \overset{k}{\rightarrowtail} X \underset{s}{\overset{f}{\rightleftarrows}} Y \longrightarrow 0$$

if X is a projective object then K is projective.

Trivial examples [CRVdL25]

- Any abelian category satisfies (P): $X \cong K \oplus Y$;
- Any Schreier variety of algebras (e.g. Gp, $\mathrm{Lie}_{\mathbb{K}}$ ($\mathbb{K}$ is a field), Ab, ...) satisfies (P): K is a subobject of a free object X.

Internal actions

How to express an action of A on X?

It can be expressed as the (**bold**) bottom split short exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A \diamond X & \rightarrowtail & A + X & \xrightarrow{\Sigma_{A,X}} & A \times X & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \langle s, \ker(f) \rangle} & & \Big\downarrow{\scriptstyle \pi_1} & & \\
\mathbf{0} & \longrightarrow & \mathbf{X} & \xrightarrow{\ker(f)} & \mathbf{Y} & \underset{s}{\overset{f}{\rightleftarrows}} & \mathbf{A} & \longrightarrow & \mathbf{0}
\end{array}
$$

where $A \diamond X$ is called the **binary cosmash product of** A **and** X.

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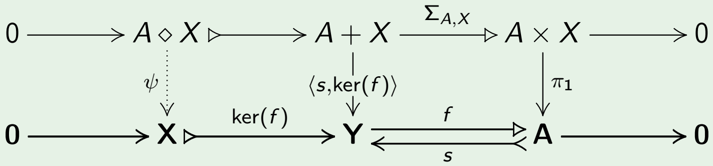

where $A \diamond X$ is called the **binary cosmash product of** A **and** X.

The original definition of **internal crossed modules** (G. Janelidze [Jan03]) is expressed in terms of an algebra over the monad $A\flat-$.
Today, I make use of "ψ" (which codifies the above split short exact sequence via a semi-direct product construction) which leads to an alternative characterization (M. Hartl and T. Van der Linden [HVdL13]).
This approach leads to shorter proofs.

Definition of internal crossed modules

Definition

A **internal crossed module** is given by
$(X \in \mathcal{C}, A \in \mathcal{C}, \partial \colon X \to A, \psi \colon A \diamond X \to X)$ where ψ is an **action core** and where ∂ is called the **boundary morphism**, satisfying three conditions.
An **internal crossed module morphism**
$(f_X, f_A) \colon (X, A, \psi, \partial) \to (X', A', \psi', \partial')$ is a pair of morphisms
$f_X \colon X \to X'$, $f_A \colon A \to A'$ in $\mathcal{C}$ compatible with the action cores and with the boundary morphisms.
This forms a category denoted $\mathrm{XMod}(\mathcal{C})$
where $\mathcal{C}$ is the underlying semi-abelian category.

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Remark

The **motivation** for the above definition is the equivalence of categories between $\mathrm{XMod}(\mathcal{C})$ and $\mathrm{Cat}(\mathcal{C})$ the category of internal categories of $\mathcal{C}$.

A projective crossed module

Proposition [CCRG02]

In XMod(Gp), if P is a projective group and Q is a projective P-group then the inclusion morphism $Q \to Q \rtimes P$ is a projective crossed module.

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Theorem [Cul25]

If P is a projective object in $\mathcal{C}$ and if the split extension

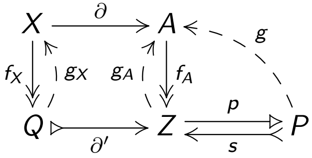
$$0 \longrightarrow Q \overset{\partial'}{\rightarrowtail} Z \underset{s}{\overset{p}{\rightleftarrows}} P \longrightarrow 0$$

is a projective object in the category of split extensions of P,
then the kernel ∂', viewed as an internal crossed module,
is a projective object in XMod($\mathcal{C}$).

Any kernel can be endowed with a (unique) crossed module structure: the action is the **conjugation action core** (denoted $\overline{\chi}$), and the boundary map is the **inclusion** ∂'.

Sketch of the proof

Consider a regular epimorphism $(f_X, f_A)\colon (X, A, \phi, \partial) \to (Q, Z, \overline{\chi}, \partial')$ in $\mathsf{XMod}(\mathcal{C})$:



1. Lifting of s along f_A (P is projective);
2. A section of f_X (the bottom is projective object in $\mathsf{SSE}_P(\mathcal{C})$);
3. A section of f_A (the construction of $Z \cong Q \rtimes_\psi P$);
4. The pair of sections is a morphism in $\mathsf{XMod}(\mathcal{C})$ ("$\diamond$" characterization).

Free crossed modules in variety $\mathcal{V}$

Consider a semi-abelian **variety of algebras** $\mathcal{V}$ with $F_r \colon \mathrm{Set} \to \mathcal{V}$ the associated free functor. All free internal crossed modules are of the form

$$(F_r(S) \flat F_r(S), F_r(S) + F_r(S), \overline{\chi}, \kappa_{F_r(S),F_r(S)})$$

where $\kappa_{F_r(S),F_r(S)} \colon F_r(S) \flat F_r(S) \to F_r(S) + F_r(S)$, for some $S \in \mathrm{Set}$.

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Corollary [Cul25]

For any non-trivial semi-abelian variety $\mathcal{V}$, the variety $\mathsf{XMod}(\mathcal{V})$ is not a Schreier variety (free objects are not stable under subobjects).

Sketch of the proof

Consider two different projectives objects P and X in $\mathcal{V}$, then

$$0 \longrightarrow P \flat X \overset{\kappa_{P,X}}{\rightarrowtail} P + X \underset{\iota_1}{\overset{\langle 1_P, 0 \rangle}{\rightleftarrows}} P \longrightarrow 0$$

the kernel part is projective in $\mathsf{XMod}(\mathcal{V})$ but not free since $P \neq X$.

Condition (P) in crossed modules

Theorem [Cul25]

Consider a semi-abelian variety $\mathcal{V}$, if $\mathcal{V}$ satisfies the condition (P), then so does the variety $\mathsf{XMod}(\mathcal{V})$.

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Comments

The proof relies on

- another characterization of (P) expressed in terms of free objects in Mal'tsev variety [CRVdL25];
- and it is also based on the two main results previously explained today!

Thank you!

Questions? Or comments?

References

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