Schemes relative to Actegories

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Usual Algebraic Geometry on the big site of affine schemes	Relative Algebraic Geometry [Toën and Vaquié, 2009]
$(\mathrm{Ab}, \otimes, \mathbb{Z})$	$(\mathcal{C}, \otimes, 1)$
CRing	$\mathrm{Comm}(\mathcal{C})$ (commutative monoids in $\mathcal{C}$)
Zariski Site : AffSch	"Zariski" Site : $\mathrm{Aff}_{\mathcal{C}} := \mathrm{Comm}(\mathcal{C})^{op}$ (affines)
Schemes	$\mathcal{C}$ — Schemes

- Toën and Vaquié used actions of commutative monoids in $\mathcal{C}$ <u>on objects of $\mathcal{C}$</u> to define a Grothendieck topology on $\mathrm{Aff}_{\mathcal{C}} = \mathrm{Comm}(\mathcal{C})^{op}$.

Microcosm Principle : Categorification allows Internalization

(symmetric) Monoidal Categories — Actegories

Vertical Categorification

(commutative) monoids — Actions

In this work : $(\mathcal{C}, \otimes, 1)$ + $(\mathcal{M}, \boxtimes)$

Bénabou cosmos

left $\mathcal{C}$- actegory

bicomplete , $\mathcal{C} \times \mathcal{M} \xrightarrow{\boxtimes} \mathcal{M}$

cocontinuous in both arguments

$\mathrm{Comm}(\mathcal{C})$ $\longrightarrow$ "$\mathcal{M}$ - Zariski" Site $\longrightarrow$ $\mathcal{M}$ - Schemes

(comm. monoids in $\mathcal{C}$)

$\mathrm{Aff}_{\mathcal{C}} = \mathrm{Comm}(\mathcal{C})^{op}$

(affines)

- $c \in \mathrm{Comm}(\mathcal{C}) \implies$ monad $c \boxtimes -$ on $\mathcal{M}$:

$$\mathrm{Mod}_{\mathcal{M}}(c) := \mathcal{M}^{c \boxtimes -}$$

Eilenberg-Moore category

(actions of c in $\mathcal{M}$)

- $\mathrm{Mod}_{\mathcal{M}}(c) \xrightarrow{\text{forgetful}} \mathcal{M}$ creates limits and colimits

$\implies$ $\mathrm{Mod}_{\mathcal{M}}(c)$ is bicomplete

$\alpha^* =$ Extension

$\implies$ $\alpha : a \longrightarrow b$ in $\mathrm{Comm}(\mathcal{C})$ induces:

$$\begin{tikzcd} \mathrm{Mod}_{\mathcal{M}}(a) \arrow[r, bend left, "\alpha^*"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_{\mathcal{M}}(b) \arrow[l, bend left, "\alpha_*"] \end{tikzcd}$$

$\alpha_* =$ Restriction

The construction

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, "\alpha"'] & \mathrm{Mod}_{\mathcal{M}}(a) \arrow[d, "\alpha^*"] \\ {} \arrow[r, phantom, "\longmapsto"] & {} \\ b \arrow[r, mapsto] & \mathrm{Mod}_{\mathcal{M}}(b) \end{tikzcd}$$

α^* = extension of scalars

defines a pseudo-functor $\mathrm{Comm}(\mathcal{C}) \longrightarrow \mathrm{Cat}$

Notation : $\mathrm{Comm}(\mathcal{C}) \ni a \quad \rightsquigarrow \quad \mathrm{Spec}(a) \in \mathrm{Aff}_{\mathcal{C}} = \mathrm{Comm}(\mathcal{C})^{op}$

(affine)

Defn. : $\left\{ \mathrm{Spec}(a_i) \xrightarrow{\alpha_i^{op}} \mathrm{Spec}(a) \right\}_{i \in I}$ Zariski $\mathcal{M}$-cover

* if $\forall\, i \in I$, $\alpha_i : a \longrightarrow a_i$ in $\mathrm{Comm}(\mathcal{C})$ is an

$\mathcal{M}$-flat + epimorphism + of finite type

$\mathrm{Mod}_{\mathcal{M}}(a) \xrightarrow{\alpha_i^*} \mathrm{Mod}_{\mathcal{M}}(a_i)$ is left-exact

${}^{a}/_{\mathrm{Comm}(\mathcal{C})}(\alpha_i, -) : \mathrm{Comm}(\mathcal{C}) \longrightarrow \mathrm{Set}$ preserves filtered colimits

* $\exists\, K \underset{\text{finite}}{\subseteq} I$ such that $\{\alpha_k^*\}_{k \in K}$ is collectively conservative.

Zariski $\mathcal{M}$-site : $(\mathrm{Aff}_{\mathcal{C}}, J_{\mathcal{M}})$ where $J_{\mathcal{M}}$ has as basis :

$$\mathrm{Aff}_{\mathcal{C}} \ni \mathrm{Spec}(a) \longmapsto \left\{ \begin{array}{c} \text{Zariski } \mathcal{M}\text{-covers for} \\ \mathrm{Spec}(a) \end{array} \right\}$$

- In particular, $(\mathcal{M}, \boxtimes) := (\mathcal{C}, \otimes)$ recovers Toën and Vaquié's theory.

- $J_{\mathcal{M}}$ is not always subcanonical!
  Call $\mathcal{M}$ to be **subcanonical** if $J_{\mathcal{M}}$ is subcanonical.

Example

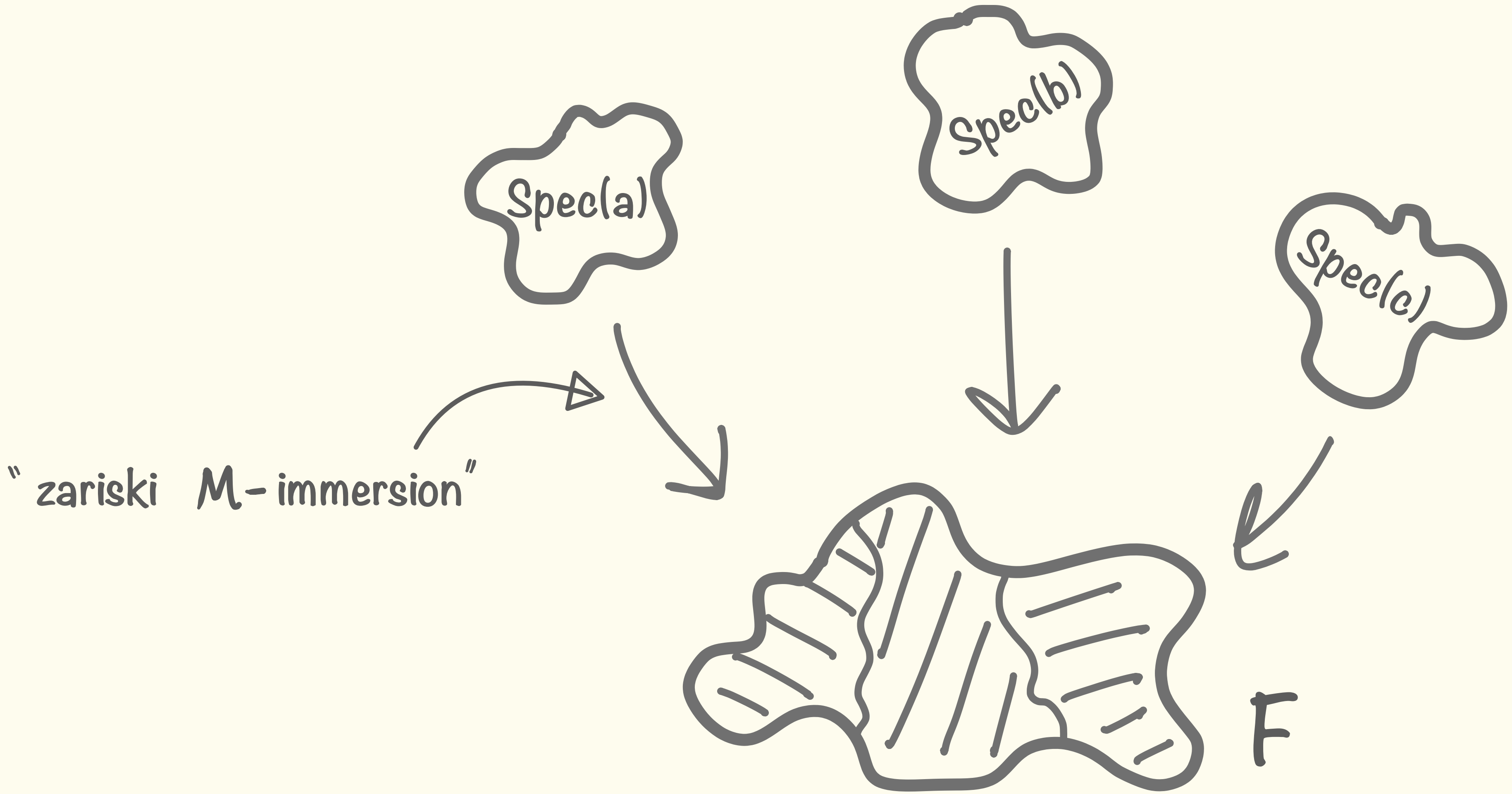
cartesian monoidal category of directed-graphs $\curvearrowright$ $(\mathrm{Digrph}, \times, 1)$

The $(\mathrm{Digrph}, \times, 1)$-actegory structure on $(\mathrm{Digrph}, \times)$ restricts under

$$\mathrm{Set} \xrightarrow{\text{disc}} \mathrm{Cat} \xrightarrow{\text{forgetful}} \mathrm{Digrph}$$

to make Digrph into a subcanonical $(\mathrm{Set}, \times, 1)$-actegory.

M - scheme : Sheaf $F : \mathrm{Aff}_{\mathcal{C}}^{op} \longrightarrow \mathrm{Set}$ w.r.t J_M which can be "nicely covered" by representables in $\mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}}, J_M)$.



$\mathrm{Sch}(\mathcal{C})_{\mathcal{M}}$: full subcategory of $\mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}}, J_{\mathcal{M}})$ consisting of $\mathcal{M}$-schemes.

- $\mathrm{Sch}(\mathcal{C})_{\mathcal{M}} \hookrightarrow \mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}}, J_{\mathcal{M}})$ is closed under pullbacks, coproducts, quotients by "nice" equivalence relations.

- $$\begin{tikzcd}
\mathrm{Aff}_{\mathcal{C}} \arrow[r, hook] \arrow[rr, bend left, "\text{yoneda}"] \arrow[dr, dashed] & \mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}}, J_{\mathcal{M}}) \arrow[r, hook] & \mathrm{PSh}(\mathrm{Aff}_{\mathcal{C}}) \\
& \mathrm{Sch}(\mathcal{C})_{\mathcal{M}} \arrow[u, hook] \arrow[ul, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

Change of Base

- an adjunction in $\mathrm{symMonCat}_{\mathrm{lax}}$:
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "B"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "A"] \end{tikzcd}$$
induces adjunctions of $\mathcal{C}$omm's and Aff's

- $\mathcal{C}$-actegory $\mathcal{M}$, $\mathcal{D}$-actegory $\mathcal{N}$ [not necessarily subcanonical]

[B is strong monoidal $\Rightarrow$ $\mathcal{D}$-action on $\mathcal{N}$ restricts along B to a $\mathcal{C}$-action on $\mathcal{N}$]

- a lax $\mathcal{C}$-linear functor $L : \mathcal{N} \longrightarrow \mathcal{M}$

Theorem : If

① $A : \mathcal{D} \longrightarrow \mathcal{C}$ preserves filtered colimits,

② $L : \mathcal{N} \longrightarrow \mathcal{M}$ is conservative and left-exact,

③ a $\boxed{\text{technical condition}}$ holds, then,

1. the functor $\_\circ B^{op} : PSh(Aff_{\mathcal{D}}) \longrightarrow PSh(Aff_{\mathcal{C}})$ restricts to a functor $B_! : Sh(Aff_{\mathcal{D}}, J_N) \longrightarrow Sh(Aff_{\mathcal{C}}, J_M)$ which has a left-adjoint $A_!$.

2. If M and N are subcanonical, the left-adjoint $A_!$ restricts to a functor $Sch(\mathcal{C})_M \longrightarrow Sch(\mathcal{D})_N$

such that the following diagram commutes upto a natural isomorphism:

$$
\begin{tikzcd}
\mathrm{Sch}(\mathcal{C})_{\mathcal{M}} \arrow[r, "A_!"] \arrow[dr, phantom, "\cong"] & \mathrm{Sch}(\mathcal{D})_{\mathcal{N}} \\
\mathrm{Aff}_{\mathcal{C}} \arrow[u, hook] \arrow[r, "B^{op}"'] & \mathrm{Aff}_{\mathcal{D}} \arrow[u, hook]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Comm}(\mathcal{C}) \arrow[r, "B"] & \mathrm{Comm}(\mathcal{D})
\end{tikzcd}
$$

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Thank You