Extending an arithmetic universe by an object

By Samuel Desrochers (he/him) – University of Ottawa

Under the supervision of Simon Henry and Phil Scott

$$\mathcal{A} \longrightarrow \mathcal{A}[X]$$

Categorical logic

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In categorical logic, we study ***geometric morphisms*** *between* ***elementary toposes***

$$F : \mathcal{S} \to \mathcal{C}$$

Note – for this presentation, I represent geometric morphisms with the left adjoint functor.

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In logic, we study theories, for example:

Signature:
- A type G
- Function symbols $\ast : G \times G \to G$ and $(-)^{-1} : G \to G$
- A constant e

Axioms:
- $(a \ast b) \ast c = a \ast (b \ast c)$
- $a \ast e = a = e \ast a$
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To interpret a first-order theory, we use:

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- $\bigvee_{n \in \mathbb{N}} (x^n = e)$

This statement is indexed by an infinite set

To interpret a geometric theory, we use:

- An **elementary topos** $\mathcal{C}$
- A **geometric morphism** $\boldsymbol{Set} \to \mathcal{C}$

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This leads to a theory of the **classifying topos** $\mathcal{S}[T]$ for an $\mathcal{S}$-theory T.

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The problem: geometric morphisms don't preserve all the structure of elementary toposes.

Instead: Use arithmetic universes!

$$F : \mathcal{A} \to \mathcal{B}$$

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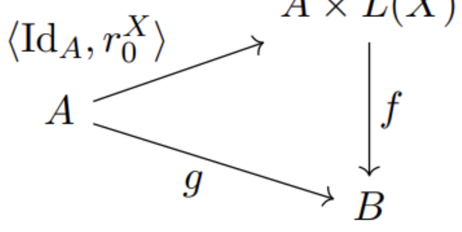
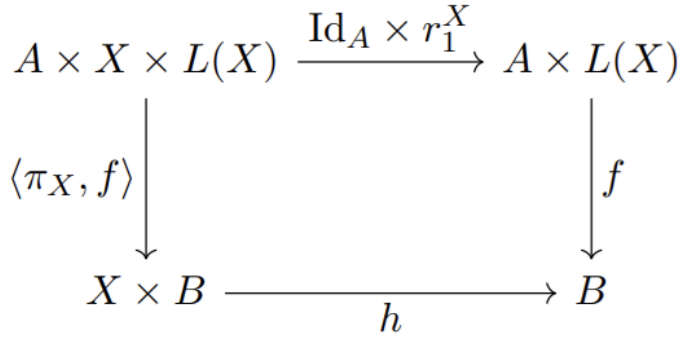
- Stable disjoint finite coproducts
- Stable effective quotients of equivalence relations
- Parametrized list objects





$f(a, \emptyset) = g(a)$

$f(a, x :: \ell) = h\big(x, f(a, \ell)\big)$

Arithmetic universes

Some properties of arithmetic universes…

- Arithmetic universes have *natural numbers objects*

$$N = L(1)$$

$$1 \xrightarrow{0} L(1)$$

$$\begin{array}{ccc} 1 \times L(1) & \longrightarrow & L(1) \\ (-,n) & \mapsto & 1+n \end{array}$$

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- Arithmetic universes have all finite colimits (**all** coequalizers!)

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$$L(X) = \sum_{n \in N} X^n$$

S. Desrochers. "**The List Object Endofunctor is Polynomial**". *arXiv:2503.20671*, 2025.

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- There is an *initial* arithmetic universe

Categorical logic with arithmetic universes

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The *classifying topos* $\mathcal{S}[T]$ for a theory T over a base category $\mathcal{S}$.

$$\left\{\begin{matrix}\text{Models of } T \\ \text{in } \mathcal{S} \to \mathcal{C}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Geometric morphisms} \\ \mathcal{S}[T] \quad \to \quad \mathcal{C} \\ \nwarrow \quad \nearrow \\ \mathcal{S}\end{matrix}\right\}$$

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Can we construct a "classifying arithmetic universe" $\mathcal{A}[T]$ for a theory T over an arithmetic universe $\mathcal{A}$?

$$\begin{Bmatrix}\text{Models of } T \\ \text{in } \mathcal{A} \to \mathcal{B}\end{Bmatrix} \longleftrightarrow \begin{Bmatrix}\text{AU morphisms} \\ \mathcal{A}[T] \quad \to \quad \mathcal{B} \\ \nwarrow \quad \nearrow \\ \mathcal{A}\end{Bmatrix}$$

Categorical logic with arithmetic universes

One of the simplest theories is $\mathbb{O}$, the theory of objects:

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- A type X

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- None

Models of $\mathbb{O}$ in $\mathcal{B}$ are precisely the objects of $\mathcal{B}$.

What is $\mathcal{A}[\mathbb{O}]$?

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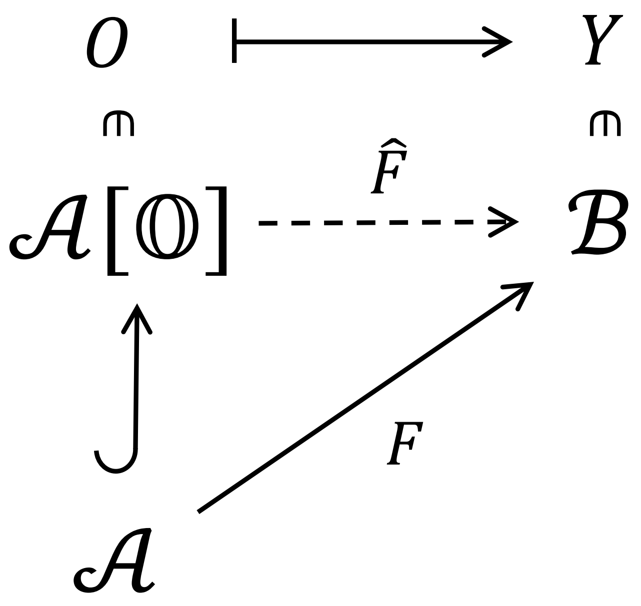
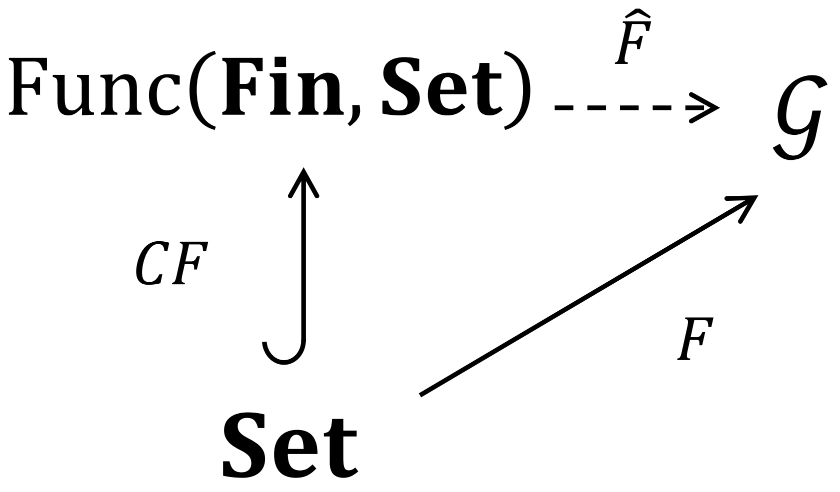


What is $\mathcal{A}[\mathbb{O}]$?

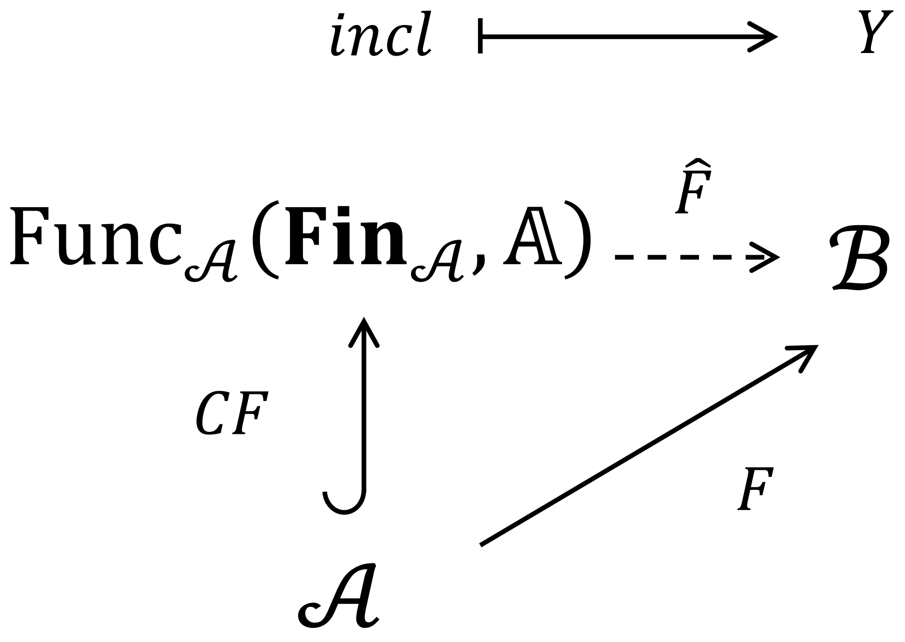
The answer is analogous to the case for Grothendieck toposes: $\mathbf{Set}[\mathbb{O}] \cong \mathrm{Func}(\mathbf{Fin}, \mathbf{Set})$.

$$incl \longmapsto Y$$



What is $\mathcal{A}[\mathbb{O}]$?

We claim that $\mathcal{A}[\mathbb{O}] \cong \mathrm{Func}_{\mathcal{A}}(\mathbf{Fin}_{\mathcal{A}}, \mathbb{A})$.



In spirit, the proof is the same. However, filling in the details requires a lot of machinery from the theory of indexed categories and arithmetic universes.

Thank you for listening!

Special thanks to Simon Henry for guiding me through this research project

Notable references

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Internal category of finite sets

- The *object of objects* for **Fin** is $C_0 = N$
  - So the "objects" of **Fin** are natural numbers
- The *object of arrows* $C_1 \rightrightarrows N$ for **Fin** is constructed as follows
  - We set $C_1 = \{b^a : a, b \in N\}$
  - Formally: set $E = \{k, n \in N \mid k < n\}$.
    Then $\pi_2^E : E \to N$ satisfies $(\pi_2^E)^{-1}(n) = \{0, \dots, n-1\}$.
    $C_1 \rightrightarrows N$ is an exponential in $\mathcal{A}/N \times N$ of $E \times N \to N \times N$ and $N \times E \to N \times N$
- Composition $C_1 \times_N C_1 \to C_1$ and identity $N \to C_1$ are straightforward