Call doctrines by your name

Ivan Di Liberti

CT25
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    - what's a *doctrine* in categorical logic?
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2. Kan injectivity and semantic prescriptions
3. Syntactic categories and syntactic sites
4. Kock-Zoberlein doctrines on $\mathbf{Lex}$
5. Classifying topoi and Diaconescu
6. Completeness theorems and open problems
7. From Kock-Zoberlein doctrines on $\mathbf{Lex}$ to Lawvererian doctrines

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In the recent paper **KZ monads and Kan Injectivity** by Sousa, Lobbia and DL this behaviour is called Weak Kan Injectivity (with respect to a morphism f).

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For $\mathcal{H}$ a logic, consider the composition below.

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The last result hinges on Makkai's conceptual completeness and we do not have a non-semantic proof of this result.

Construction: the classifying topos of an algebra

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Every algebra can be equipped with a canonical structure of site, on which we can take sheaves.

$$\begin{tikzcd}
\mathsf{Alg}(\mathsf{T}^{\mathcal{H}})_{\mathsf{M}} \arrow[rr, "\mathsf{Cl}"] \arrow[dr, swap, "J^{\mathcal{H}}"] & & \mathsf{Topoi}^{\mathsf{op}} \\
& \mathsf{SITES}_{\mathsf{M}} \arrow[ur, swap, "\mathsf{Sh}"] &
\end{tikzcd}$$

Theoremm: Diaconescu

We have a relative pseudoadjunction as below,

$$\begin{tikzcd}
\mathsf{alg}(\mathsf{T}^{\mathcal{H}}) \arrow[r, "\mathsf{Cl}"] \arrow[r, phantom, "\perp"', shift right=3] \arrow[d] & \mathsf{Topoi}^{\mathsf{op}} \arrow[dl] \\
\mathsf{Alg}(\mathsf{T}^{\mathcal{H}}) &
\end{tikzcd}$$

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\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{lex} \arrow[r, "\mathrm{Cl}", "\perp"'] \arrow[d] & \mathrm{Topoi}^{\mathrm{op}} \arrow[dl] \\
\mathrm{LEX} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Topoi}^{\mathrm{op}} \arrow[r, "\mathrm{Cl}", "\perp"'] \arrow[d] & \mathrm{Topoi}^{\mathrm{op}} \arrow[dl] \\
\mathrm{Pretopoi}_{\infty} &
\end{tikzcd}$$

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Of course this theorem ought to be true, but until recently we did not even have the language to state (especially in categorical language).

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