Call doctrines by your name

Ivan Di Liberti CT25 July 2025, Brno.



This talk is based on a preprint and an ongoing project.

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Motivations:



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 - what's a doctrine in categorical logic?
 - what's a fragment of geometric logic?
- Kan injectivity and semantic prescriptions
- Syntactic categories and syntactic sites
- 4 Kock-Zoberlein doctrines on Lex
- Classifying topoi and Diaconescu
- 6 Completeness theorems and open problems
- From Kock-Zoberlein doctrines on Lex to Lawvererian doctrines







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(Weak Kan Injectivity)

In the recent paper **KZ monads and Kan Injectivity** by Sousa, Lobbia and DL this behaviour is called Weak Kan Injectivity (with respect to a morphism f).





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For \mathcal{H} a logic, consider the composition below.

$$\text{lex} \xrightarrow{\mathsf{Psh}} \mathsf{WRInj}(\mathcal{H})^{\mathsf{op}} \xrightarrow{\mathsf{Syn}^{\mathcal{H}}} \mathsf{LEX}$$



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Achtung!

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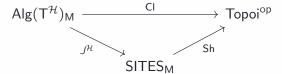


Construction: the classifying topos of an algebra



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Every algebra can be equipped with a canonical structure of site, on which we can take sheaves.





Theoremm: Diaconescu

We have a relative pseudoadjunction as below,

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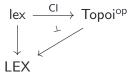
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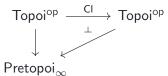
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A logic $\mathcal H$ enjoys conceptual completeness if the 2-functor exhibiting conceptual soundness $Alg(T^{\mathcal H})^{op} \to WRInj(\mathcal H)$ is in fact 2-fully faithful.

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What logics \mathcal{H} enjoy conceptual completeness?



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Of course this theorem ought to be true, but until recently we did not even have the language to state (especially in categorical language).





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