Cartesian monoidality of the cubical Joyal model structure

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$\mathrm{cSet} = \mathrm{Set}^{\square^{\mathrm{op}}}$.

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This is **not** the Cartesian product in cSet – it's not even symmetric.

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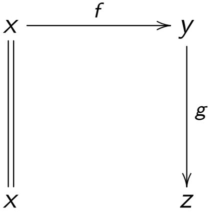

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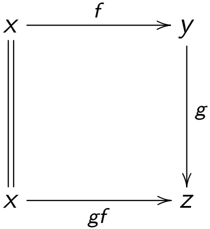

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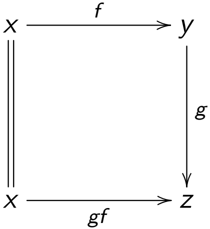

Theorem (D.-Kapulkin-Lindsey-Sattler)

cSet *carries a model structure for* $(\infty, 1)$*-categories, the* **cubical Joyal model structure***, with cubical quasicategories as fibrant objects. This model structure is monoidal with respect to the geometric product.*

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Idea: obtain P^n from $\square^n$ by repeated pushouts of inner open box fillings (anodyne maps).

Proving the trivial cofibration

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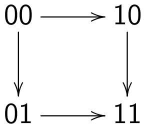

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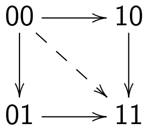

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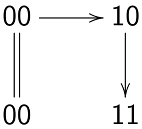

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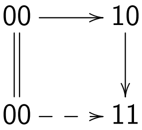
Standard decomposition cubes



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while this cube $[1]^2 \to [1]^2$ is:

$$(a, b) \mapsto (a, a \wedge b)$$

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For any $X, Y \in \mathrm{cSet}$, the natural map $X \otimes Y \hookrightarrow X \times Y$ is a trivial cofibration in the cubical Joyal model structure. $\square$

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This, together with monoidality with respect to $\otimes$, lets us show cartesian monoidality.

References

D., Symmetry in the cubical Joyal model structure. To appear in Algebr. Geom. Topol., 2024. arXiv:2409.13842

D., Krzysztof Kapulkin, Zachery Lindsey, and Christian Sattler. Cubical models of $(\infty,1)$-categories. Mem. Amer. Math. Soc., 297 (2024), no. 1484, v+110 pp. arXiv:2005.04853