An Adjunction between Categories of Monads

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PUSHFORWARD MONADS

$$\mathcal{C} \xrightarrow{\quad F \quad} \mathcal{D}$$

$$\text{monads on } \mathcal{C} \overset{F_\#}{\rightsquigarrow} \text{monads on } \mathcal{D}$$

Pushforward monads

$$\mathcal{C} \xrightarrow{\quad F \quad} \mathcal{D}$$

$$\text{monads on } \mathcal{C} \overset{F_{\#}}{\rightsquigarrow} \text{monads on } \mathcal{D}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "T"'] \arrow[dr, phantom, "\swArrow_{\varepsilon}"] & \mathcal{D} \arrow[d, "\mathrm{Ran}_F FT"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

Pushforward monads

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\text{monads on } \mathcal{C} \overset{F_\#}{\rightsquigarrow} \text{monads on } \mathcal{D}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "T"'] & \mathcal{D} \arrow[d, "\mathrm{Ran}_F FT"] \\ \mathcal{C} \arrow[r, "F"'] \arrow[ur, phantom, "\Swarrow \varepsilon"] & \mathcal{D} \end{tikzcd} \quad \mathrm{Ran}_F FT \; \boxed{=: F_\# T}$$

"the pushforward of T along F"

[Street]

Pushforward Monads

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\text{monads on } \mathcal{C} \xrightarrow{F_\#} \text{monads on } \mathcal{D}$$

$$T \longmapsto \boxed{F_\# T = \mathrm{Ran}_F\, FT}$$

PUSHFORWARD MONADS

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$$T \longmapsto \boxed{F_\# T = \mathrm{Ran}_F\, FT}$$

- If $G \dashv F$, then $F_\# T = FTG$.

Pushforward Monads

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\text{monads on } \mathcal{C} \overset{F_\#}{\rightsquigarrow} \text{monads on } \mathcal{D}$$

$$T \longmapsto \boxed{F_\# T = \mathrm{Ran}_F FT}$$

- If $G \dashv F$, then $F_\# T = FTG$.
- If $T = 1_{\mathcal{C}}$, then $F_\# T$ is the codensity monad of F.

Pushforward as a Functor

Universal property of $F_\# T$

[Street]

$$\begin{tikzcd} \mathcal{C}^T \arrow[r, "K^{F,T}"'] \arrow[d, "U^T"'] & \mathcal{D}^{F_\# T} \arrow[d, "U^{F_\# T}"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

Pushforward as a functor

Universal property
of $F_{\#}T$
[Street]

$$
\begin{tikzcd}
 & & \forall\, \mathcal{D}^{S} \arrow[ddl, bend left, "U^{S}"] \\
\mathcal{C}^{T} \arrow[rru, bend left, "\forall\, \bar{F}"] \arrow[r, "K^{F,T}"'] \arrow[d, "U^{T}"'] & \mathcal{D}^{F_{\#}T} \arrow[ru, dashed, "\exists!\, U"] \arrow[d, "U^{F_{\#}T}"] & \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D} &
\end{tikzcd}
$$

Pushforward as a functor

Universal property of $F_\# T$

[Street]

$$\begin{tikzcd}
\mathcal{C}^T \arrow[r, "K^{F,T}"'] \arrow[d, "U^T"'] \arrow[rr, bend left, "\forall \bar{F}"] & \mathcal{D}^{F_\# T} \arrow[d, "U^{F_\# T}"] \arrow[r, dashed, "\exists ! U"] & \forall \mathcal{D}^S \arrow[dl, bend left, "U^S"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D} &
\end{tikzcd}$$

representably small $\rightarrow$ $\mathcal{C} \xrightarrow{F} \mathcal{D}$ $\leftarrow$ complete

$$\mathrm{Mnd}_{\mathcal{C}} \xrightarrow{F_\#} \mathrm{Mnd}_{\mathcal{D}}$$

PUSHFORWARD AS A RIGHT ADJOINT

representably small →

fully faithful

$$\mathcal{C} \overset{F}{\hookrightarrow} \mathcal{D}$$

← complete

$$\mathrm{Mnd}_{\mathcal{C}} \overset{F_\#}{\hookrightarrow} \mathrm{Mnd}_{\mathcal{D}}$$

[DM]

PUSHFORWARD AS A RIGHT ADJOINT

representably small $\rightarrow$ $\mathcal{C} \overset{F}{\hookrightarrow} \mathcal{D}$ $\leftarrow$ complete

fully faithful

$$\begin{tikzcd} \mathrm{Mnd}_{\mathcal{C}} \arrow[r, bend right, hook, "F_{\#}"'] & \mathrm{Mnd}_{\mathcal{D}}^{\mathrm{res}\,\mathcal{C}} \arrow[r, hook] & \mathrm{Mnd}_{\mathcal{D}} \end{tikzcd}$$

monads that fix $\mathcal{C}$ up to iso

[DM]

Pushforward as a Right Adjoint

representably small → ← fully faithful

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "F"] & \mathcal{D} \end{tikzcd}$$

← complete

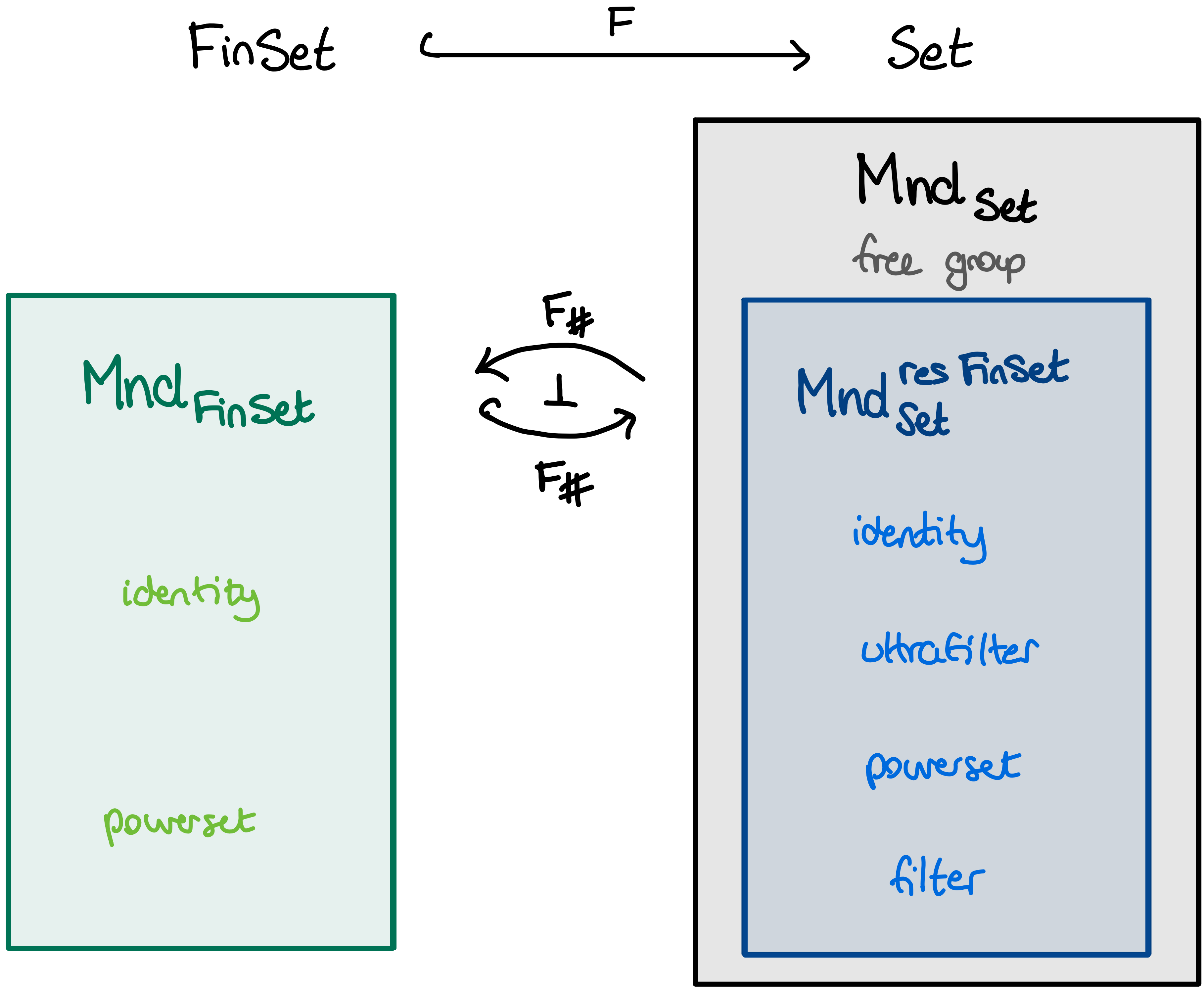
restrict to a monad on $\mathcal{C}$

$$\begin{tikzcd} \mathrm{Mnd}_{\mathcal{C}} \arrow[r, bend right, "F_{\#}"'] \arrow[r, phantom, "\perp"] & \mathrm{Mnd}_{\mathcal{D}}^{\mathrm{res}\,\mathcal{C}} \arrow[l, bend right, "F^{\#}"'] \arrow[r, hook] & \mathrm{Mnd}_{\mathcal{D}} \end{tikzcd}$$

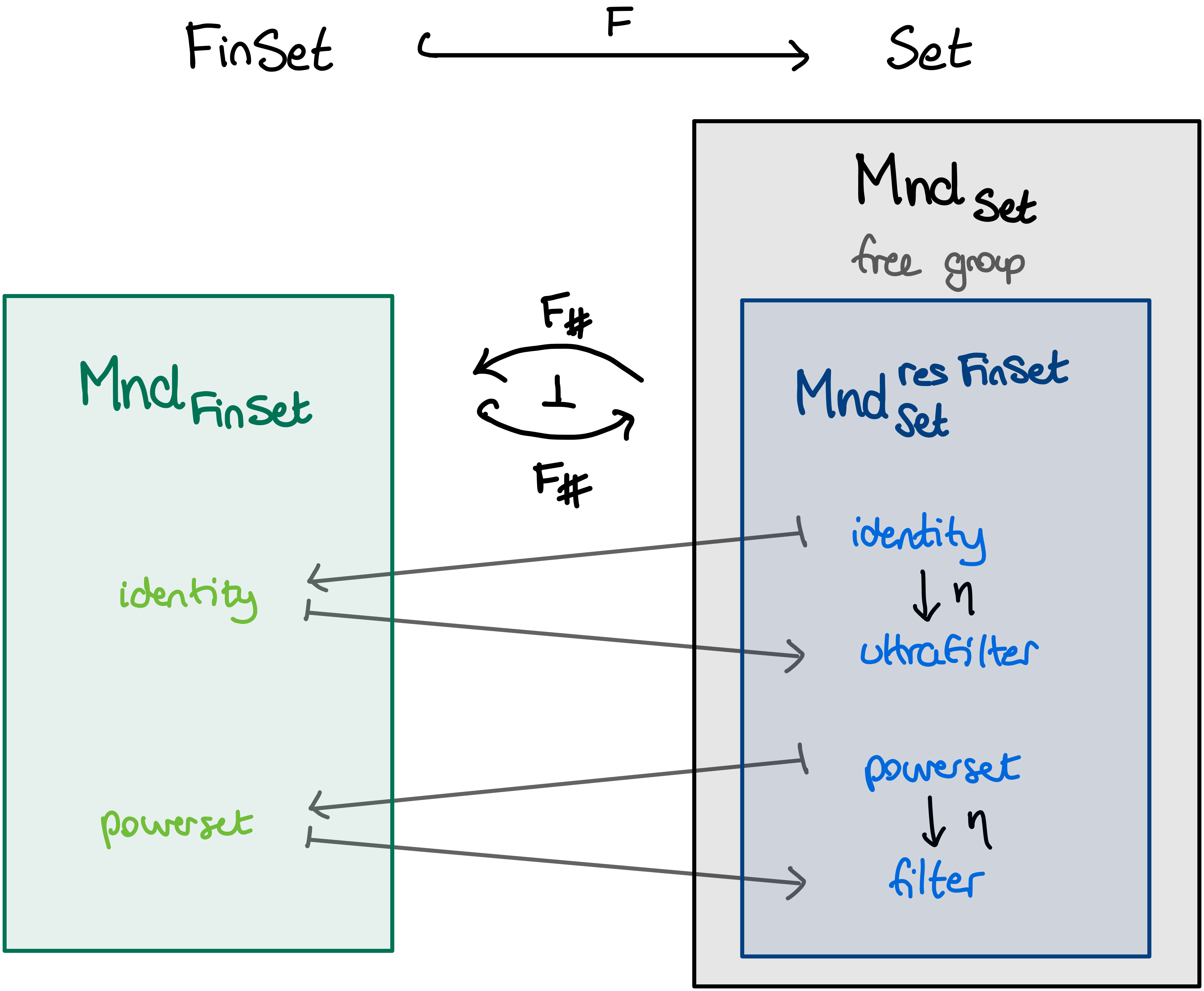
↑

monads that fix $\mathcal{C}$ up to iso

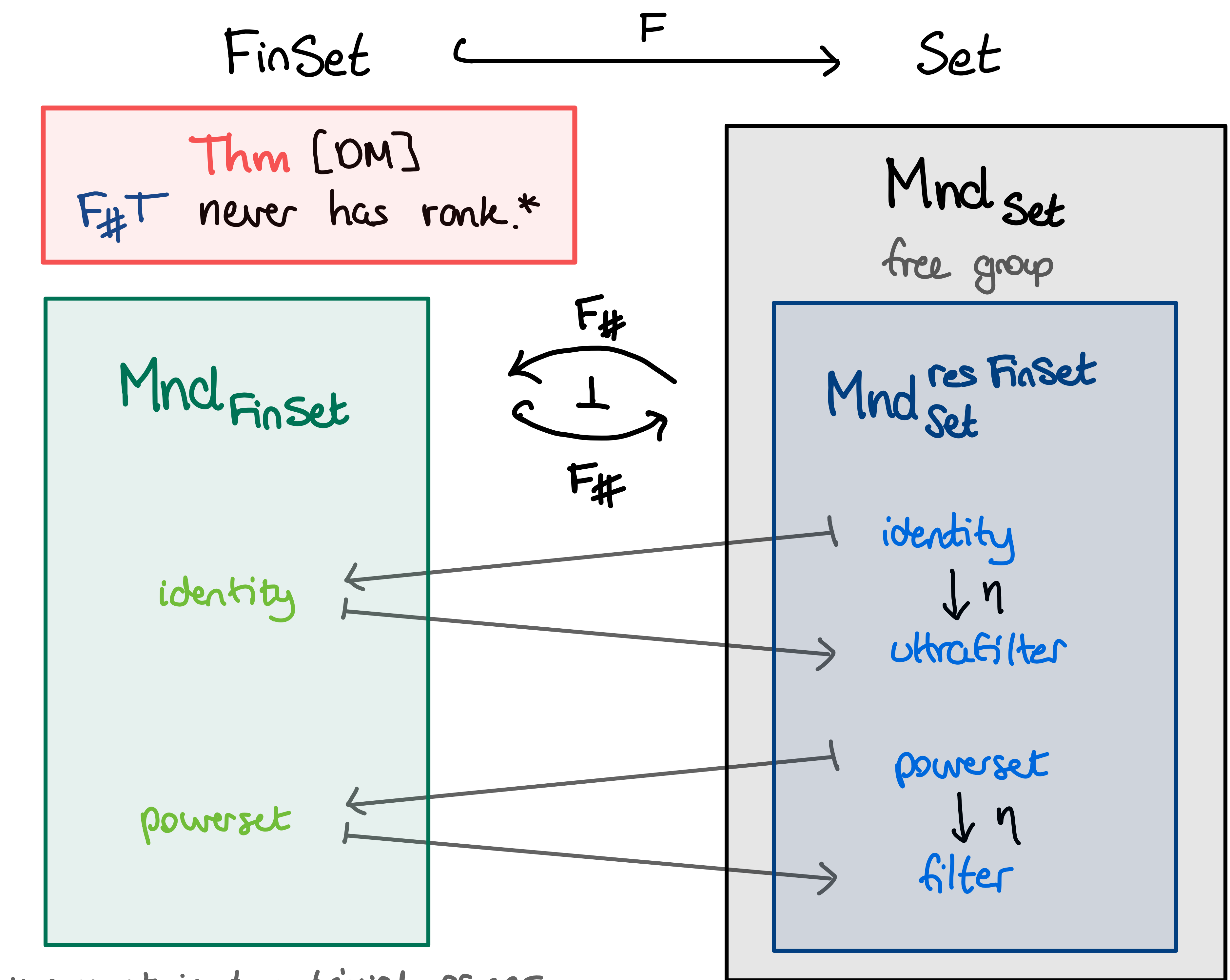
[DM]



FinSet
F
Set
$\mathrm{Mnd}_{\mathrm{Set}}$
free group
$\mathrm{Mnd}_{\mathrm{FinSet}}$
identity
powerset
$F_\#$
$\perp$
$F_\#$
$\mathrm{Mnd}^{\mathrm{res\ FinSet}}_{\mathrm{Set}}$
identity
ultrafilter
powerset
filter



FinSet
F
Set
Mnd Set
free group
Mnd FinSet
F#
⊥
F#
Mnd res FinSet Set
identity
η
ultrafilter
powerset
η
filter
identity
powerset



* except in two trivial cases.

Thank you

References

[Street] R. Street, The formal theory of monads

[DM] A. Doña Mateo, Pushforward monads