The monad of pushforwards and its decomposition

Arnaud Duvieusart

UCLouvain - FNRS

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Pushforwards of short exact sequences

Proposition (Cigoli, Mantovani, Metere)

Given a short exact sequence $K \xrightarrow{k} X \xrightarrow{q} Q$*, an action* $\xi \colon X\flat L \to L$ *and an* X*-equivariant morphism* $K \xrightarrow{\varphi} L$*, there exists a morphism of short exact sequences*

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{q} & Q \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ L & \xrightarrow[l]{} & Y & \xrightarrow[q']{} & Q \end{array}$$

if and only if $(\varphi \rtimes X)^*\chi = [k, 1\rangle^*\xi \colon (K \rtimes X)\flat L \to L$.

But (when) is it enough to ask instead that $\varphi^*\chi_L = k^*\xi$?

We define

$$(S) = \quad K \xrightarrow{k} X \xrightarrow{q} Q$$

and the category $S \backslash \mathbf{SES}(\mathcal{C})$ whose objects are morphisms of short exact sequences

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{q} & Q \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ L & \xrightarrow[l]{} & Y & \xrightarrow[r]{} & R. \end{array}$$

We have a functor $U_S \colon S \backslash \mathbf{SES}(\mathcal{C}) \to \mathcal{C}$, sending the diagram above to L.

This functor U_S admits a left adjoint $F_S\colon \mathcal{C} \to S\backslash\mathbf{SES}(\mathcal{C})$, sending L to

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{q} & Q \\ \bar{i}\downarrow & & \downarrow i_X & & \| \\ T_S(L) & \xrightarrow[\sigma]{} & X+L & \xrightarrow[{[q,0]}]{} & Q. \end{array}$$

In particular, $L \mapsto T_S(L)$ defines a monad on $\mathcal{C}$.

Proposition

The category of T_S-algebra is a coreflective subcategory of $S\backslash\mathbf{SES}(\mathcal{C})$, corresponding to short exact sequences with isomorphic cokernels.

Two special cases

For

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we have $T_S(L) = K + L$; the algebras for this monad are just morphisms from K.
In general, T_S-algebras are formed by combining these two cases.

The decomposition of the monad

We have a diagram

$$
\begin{array}{ccccc}
X\flat L & \xrightarrow{\overline{\kappa}} & T_S(L) & \underset{\overline{i}}{\overset{\overline{p}}{\rightleftarrows}} & K \\
\Vert & & \downarrow{\scriptstyle \sigma} & \lrcorner & \downarrow{\scriptstyle k} \\
X\flat L & \xrightarrow{\kappa_{X,L}} & X+L & \underset{i_X}{\overset{[1,0]}{\rightleftarrows}} & X \\
\downarrow & & \downarrow{\scriptstyle [q,0]} & & \downarrow{\scriptstyle q} \\
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which shows that $T_S(L) \simeq (X\flat L) \rtimes K$.
In particular, every $S_\flat$-algebra $\xi^S\colon T_S(L) \to L$ is determined by

$$\begin{array}{ccccc}
X\flat L & \xrightarrow{\overline{\kappa}} & T_S(L) & \xleftarrow{\overline{i}} & K \\
& {\scriptstyle \xi^X_L}\searrow & \downarrow{\scriptstyle \xi^S_L} & \swarrow{\scriptstyle \varphi} & \\
& & L. & &
\end{array}$$

The decomposition of algebras

Proposition

Two morphisms $\xi_L^X \colon X \flat L \to L$ *and* $\varphi \colon K \to L$ *induce a morphism* $\xi_L^S \colon T_S(L) \to L$*, which is an* T_S*-algebra, if and only if*

1. ξ_L^X *is an* X*-action*
2. $\varphi^* \chi_L = k^* \xi_L^X \colon X \flat L \to L$
3. *the map* $[1, \varphi\rangle \colon L \rtimes K \to L$ *is* X*-equivariant.*

Comparing the results

Given an action ξ_L^X of X on L and $\varphi\colon K \to L$, there is a corresponding pushforward exact sequence if and only if

Version 1

1. φ is X-equivariant
2. $(\varphi \rtimes X)^*\chi = [k, 1\rangle^*\xi$.

Version 2

1. $\varphi^*\chi_L = k^*\xi_L^X$
2. $[1, \varphi\rangle\colon L \rtimes K \to L$ is X-equivariant.

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Both versions are equivalent to the existence of a pair of maps