Action representability of internal 2-groupoids

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Goal of the talk

1. Explain the concept of action representability that provides a common categorical description of
   - the automorphism group $\mathrm{Aut}(G)$ of a group G,
   - the Lie algebra $\mathrm{Der}(L)$ of derivations of a Lie algebra L,
   - the actor $\mathrm{Act}(\mathbb{X})$ of a crossed module $\mathbb{X}$.

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2. Explain under which conditions $\mathrm{Grpd}^2(\mathscr{C})$ and $2\text{-}\mathrm{Grpd}(\mathscr{C})$ are action representable categories.

Motivation

An **action** of a group G on a group H is a function

$$\triangleright : G \times H \to H$$

such that:

$$g \triangleright (hh') = (g \triangleright h)(g \triangleright h')$$
$$e \triangleright h = h$$
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Consequently, the functor

$$\mathrm{Act}(-, H) : \mathrm{Grp}^{\mathrm{op}} \to \mathrm{Set}, \quad G \mapsto \{G\text{-actions on } H\}$$

is representable:

$$\mathrm{Act}(-, H) \approx \mathrm{Hom}_{\mathrm{Grp}}(-, \mathrm{Aut}(H)).$$

Consider the functor:

$$\begin{aligned}\mathrm{SplExt}(-, H) : \mathrm{Grp}^{\mathrm{op}} &\to \mathrm{Set}, \\ G &\mapsto \{\text{isomorphism classes of split extensions of } G \text{ by } H\} \\ &\qquad [0 \to H \to K \rightleftarrows G \to 0]\end{aligned}$$

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We have that:

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Hence:

$$\mathrm{SplExt}(-,H) \approx \mathrm{Hom}_{\mathrm{Grp}}(-,\mathrm{Aut}(H))$$

Furthermore,

$$1_{\mathrm{Aut}(H)} \in \mathrm{Hom}_{\mathrm{Grp}}(\mathrm{Aut}(H), \mathrm{Aut}(H))$$

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satisfies the following universal property: For any split extension

$$0 \to H \to K \leftrightarrows G \to 0$$

there exist unique (up to isomorphism) morphisms φ, ψ such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H & \longrightarrow & K & \rightleftarrows & G & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \exists! \psi} & & \downarrow{\scriptstyle \exists! \varphi} & & \\
0 & \longrightarrow & H & \longrightarrow & \mathrm{Aut}(H) \rtimes H & \rightleftarrows & \mathrm{Aut}(H) & \longrightarrow & 0
\end{array}$$

commutes.

Structure of the talk

1. Semi-direct products
2. Actors
3. Split extension classifiers
4. $\mathrm{Grpd}(\mathscr{C})$
5. 2-$\mathrm{Grpd}(\mathscr{C})$

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For an abelian category $\mathscr{C}$, the functor

$$\begin{array}{ccc} \mathrm{Pt}(\mathscr{C}) & \xrightarrow{K} & \mathscr{C} \times \mathscr{C} \\ A \underset{s}{\overset{f}{\rightleftarrows}} B & \mapsto & (\mathrm{Ker}(f), B) \end{array}$$

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Proposition (Bourn; 1991)

Let $\mathscr{C}$ be a pointed category with pullbacks. TFAE:

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2. *The split short five lemma holds.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(f) & \xrightarrow{k} & A & \underset{s}{\overset{f}{\rightleftarrows}} & B \\ & & \downarrow c & & \downarrow a & & \downarrow b \\ 0 & \longrightarrow & \mathrm{Ker}(f') & \xrightarrow{k'} & A' & \underset{s'}{\overset{f'}{\rightleftarrows}} & B' \end{array}$$

b, c *isomorphisms* $\Rightarrow$ a *isomorphism*

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3. $\mathrm{Ker}_B : \mathrm{Pt}_B(\mathscr{C}) \to \mathscr{C}$, $(A, f, s) \mapsto \mathrm{Ker}(f)$ *is conservative for all objects B in $\mathscr{C}$.*

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4. $p^* : \mathrm{Pt}_B(\mathscr{C}) \to \mathrm{Pt}_E(\mathscr{C})$ *is conservative for all $p : E \to B$ in $\mathscr{C}$.*

$$\begin{array}{ccc} P & \longrightarrow & A \\ {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & \lrcorner & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ E & \xrightarrow[p]{} & B \end{array}$$

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A category $\mathscr{C}$ with pullbacks is **protomodular** if

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- Grp, $\mathrm{Grp(HComp)}$, $\mathrm{XMod(Grp)}$, Rng, Assoc_K, Lie_K, $\mathrm{Hopf}_{K,\mathrm{coc}}$

Definition (Bourn, Janelidze;1998)

A category $\mathscr{C}$ with pullbacks **has semi-direct products** if

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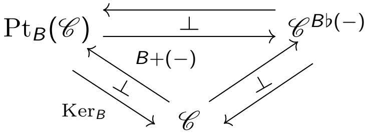
Proposition (Bourn, Janelidze; 1998)

Let $\mathscr{C}$ be a semi-abelian category. Then $\mathscr{C}$ has semi-direct products.

Let $\mathscr{C}$ be a pointed category with finite (co-)limits. Consider $\alpha_B : 0 \to B$. Then $\alpha_B^* \approx \mathrm{Ker}_B$.



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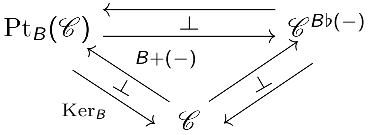

Monad

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$$A \mapsto B\flat A \xrightarrow{k_{B,A}} B + A \underset{\iota_B}{\overset{[1,0]}{\rightleftarrows}} B$$

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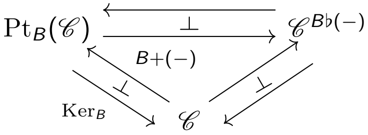

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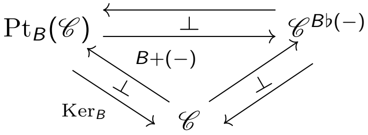

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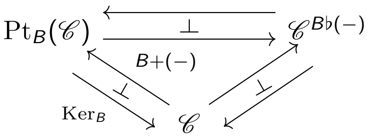

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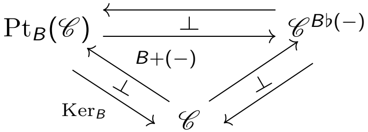

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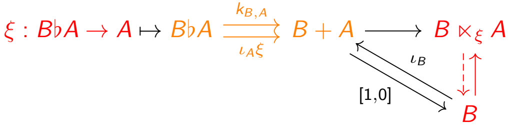

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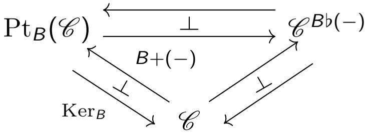

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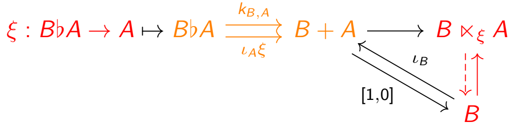

$bab^{-1} \mapsto bab^{-1}$

$\mapsto b \triangleright a$

$B \ltimes_\xi A$ is generated by B, A, with the condition that conjugation of B on A coincides with the action of B on A.

2. Actors

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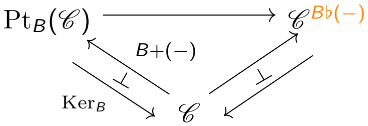
[Borceux, Janelidze, Kelly; 2005]

"Categorical algebra, understood as a categorical approach to and a categorical generalization of classical algebraic constructions [...] is still full of open questions [...] – especially those that are needed for categorical reformulations and extensions of specific group- and ring-theoretic results. Semi-abelian categories ([JMT]) provide a convenient setting for such reformulations [...]. A typical group/ring theoretic result that extends (see [BJ]) to semi-abelian categories is: Every split epimorphim is a semi-direct projection. It involves a new categorical notion of a semidirect product, and in particular a new notion of internal object action, which we continue to study in the present paper."

Let $\mathscr{C}$ be a pointed category with finite limits and finite coproducts. There is the functor

$$(-)\flat : \mathscr{C} \to \mathrm{Monad}(\mathscr{C}), \quad B \mapsto B\flat(-).$$



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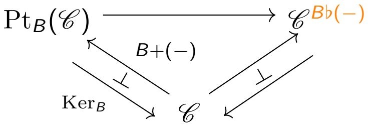

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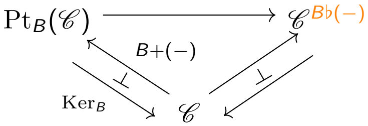

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For any object X in $\mathscr{C}$, there is the functor

$$\mathrm{Act}(-, X) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}, \quad B \mapsto \{\text{actions of } B \text{ on X}\}.$$

Definition (Borceux, Janelidze, Kelly; 2005)

Let $\mathscr{C}$ be a pointed category with finite limits and finite coproducts. $\mathscr{C}$ is **action representable** if the functor $\mathrm{Act}(-, X)$ is representable for any object X in $\mathscr{C}$:

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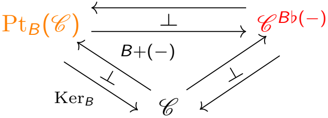

If $\mathscr{C}$ is semi-abelian, then:

$$\mathrm{Pt}_B(\mathscr{C}) \approx \mathscr{C}^{B\flat(-)}$$

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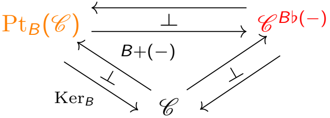

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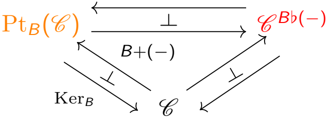

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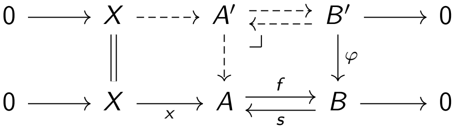
$$\begin{array}{ccl} B & \mapsto & \mathrm{SplExt}(B, X) = \{\textit{isomorphism classes of split extensions of } B \textit{ by } X\} \\ \varphi \uparrow & & \downarrow \mathrm{SplExt}(\varphi, X) \\ B' & \mapsto & \mathrm{SplExt}(B', X) \end{array}$$



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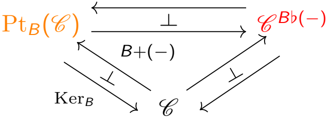

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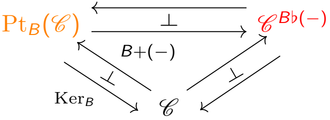

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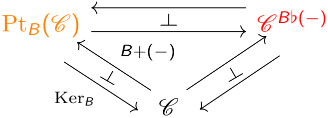

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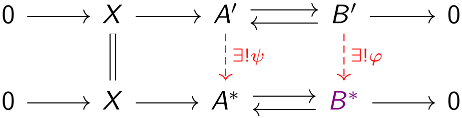

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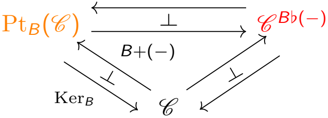

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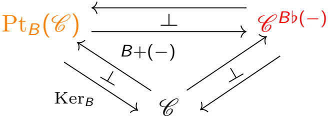

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In this case, $B^* = [X]$ *and* $A^* = [X] \ltimes_\xi X$*, where* $\xi \in \mathrm{Act}([X], X)$ *corresponds to* $1_{[X]} \in \mathrm{Hom}_{\mathscr{C}}([X], [X])$.

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- *In a semi-abelian category $\mathscr{C}$, a reflexive graph*

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- $\mathrm{Grpd}(\mathscr{C}) \approx \mathrm{Cat}(\mathscr{C})$
- $\mathrm{Grpd}(\mathscr{C})$ *is a Birkhoff subcategory of* $\mathrm{RG}(\mathscr{C})$, *i.e. a full reflective subcategory which is closed under subobjects and regular quotients.*

$$\mathrm{Grpd}(\mathscr{C}) \underset{\hookrightarrow}{\overset{\leftarrow}{\perp}} \mathrm{RG}(\mathscr{C})$$

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Examples

Grp, Lie_K, $\mathrm{Hopf}_{K,\mathrm{coc}}$.

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Let $\mathscr{C}$ *be a category which is*

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Then $\mathrm{Grpd}^2(\mathscr{C})$ *has the same properties.*

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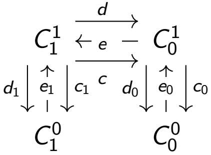

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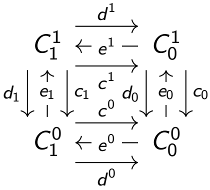
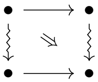


C_0^0	C_0^1	C_1^0	C_1^1
object of objects •	object of horizontal arrows • ⟶ •	object of vertical arrows • ⇝ •	object of 2-cells 

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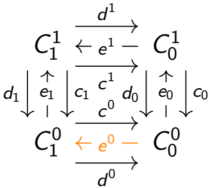

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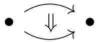
2-Grpd($\mathscr{C}$): full subcategory of $\mathrm{Grpd}^2(\mathscr{C})$ with objects such that e^0 is an isomorphism.



Main result (E., Gran)

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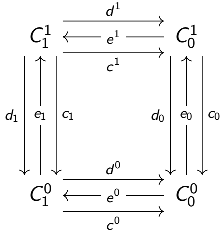

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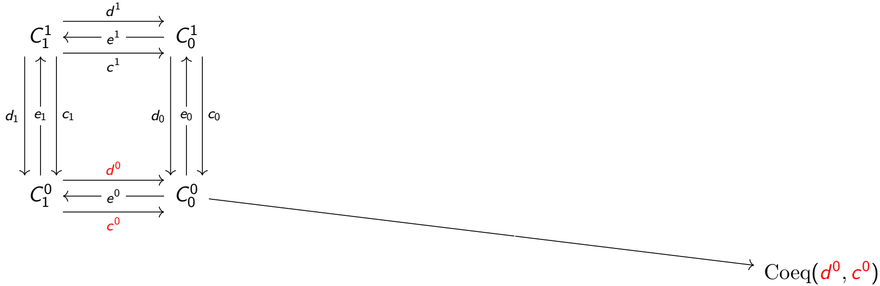

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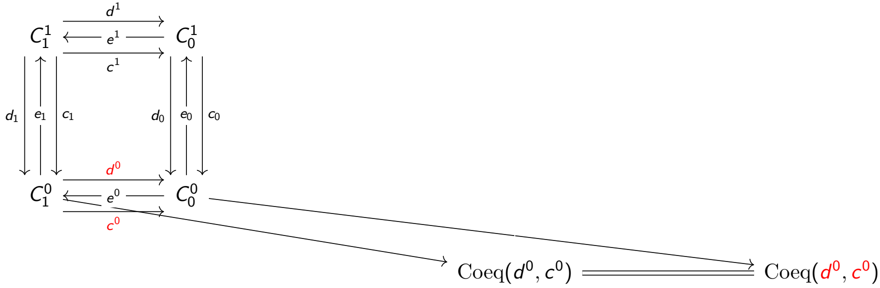

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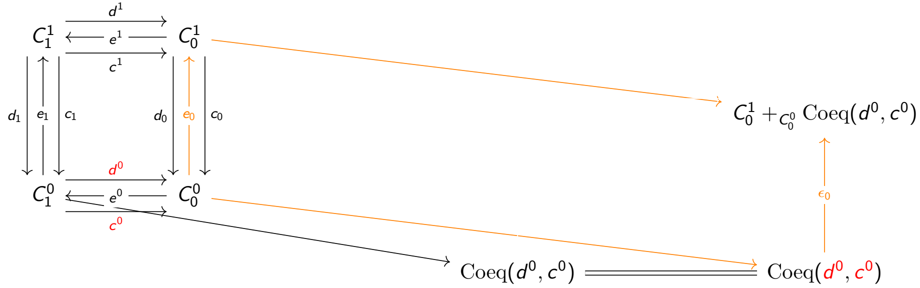

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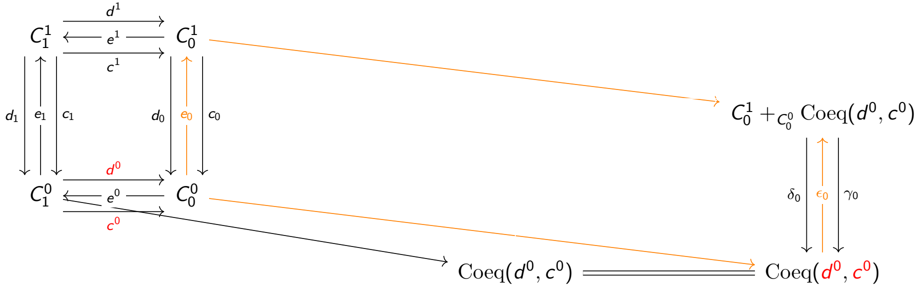

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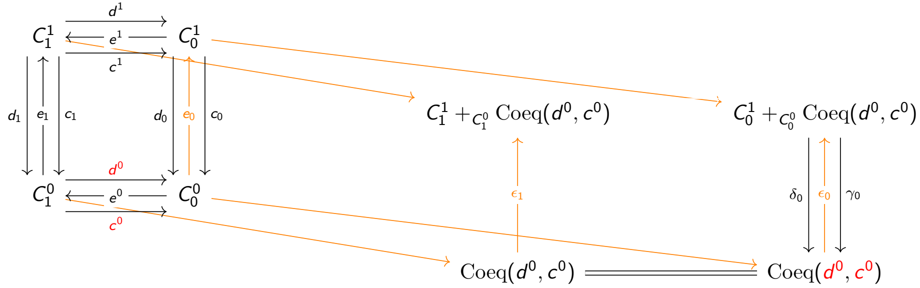

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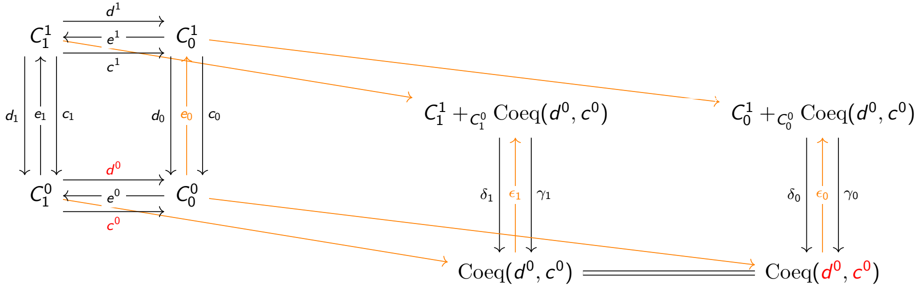

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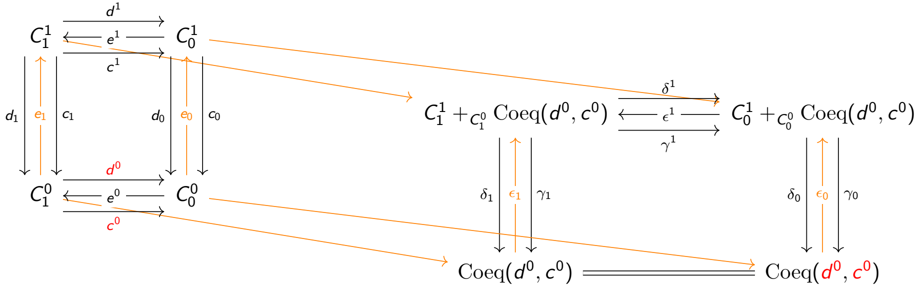

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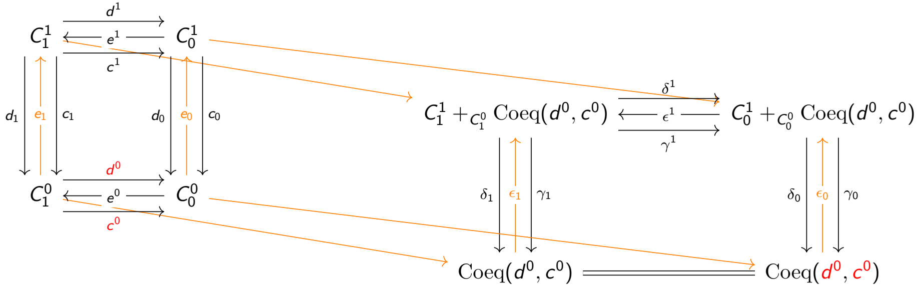

Remark

The statement is true for any regular Mal'tsev category $\mathscr{C}$ with finite colimits, such as $\mathrm{Grp}(\mathrm{Top})$, $\mathrm{Ab}_{\mathrm{t.f.}}$, Ban.

Proof of main result

- *The fact that* $2\text{-}\mathrm{Grpd}(\mathscr{C})$ *is a Birkhoff subcategory of* $\mathrm{Grpd}^2(\mathscr{C})$ *implies that* $2\text{-}\mathrm{Grpd}(\mathscr{C})$ *is semi-abelian.*

Proof of main result

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Proof of main result

- *The fact that* $2\text{-}\mathrm{Grpd}(\mathscr{C})$ *is a Birkhoff subcategory of* $\mathrm{Grpd}^2(\mathscr{C})$ *implies that* $2\text{-}\mathrm{Grpd}(\mathscr{C})$ *is semi-abelian.*
- *Moreover,* $2\text{-}\mathrm{Grpd}(\mathscr{C})$ *is a co-reflective subcategory of* $\mathrm{Grpd}^2(\mathscr{C})$*.*
- *Thus, we can use that the inclusion*

$$I : 2\text{-}\mathrm{Grpd}(\mathscr{C}) \hookrightarrow \mathrm{Grpd}^2(\mathscr{C})$$

is fully faithful, has a right adjoint and is itself a right adjoint to prove that $2\text{-}\mathrm{Grpd}(\mathscr{C})$ *has split extension classifiers.*

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