Confluence of Term Rewriting Systems with Variable Binding

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Terms with Variable Binding

Binding signature:

$$\Sigma = \left(O : \text{Set},\ |\_| : O \to \mathbb{N}^*\right)$$

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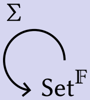

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$$\Sigma = \big(O : \mathrm{Set},\ |\_| : O \to \mathbb{N}^*\big)$$

Signature endofunctor:

$$\Sigma \circlearrowright \mathrm{Set}^{\mathbb{F}}$$

Terms over variables $V = Y(1)$:

initial $(\Sigma + V)$-algebra

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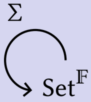

Terms in metacontext $\mathfrak{M}$:

$$\text{initial } (\Sigma + V + (\mathfrak{M} \bullet -))\text{-algebra}$$

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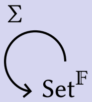

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$$T_\Sigma : \mathrm{Set}^{\mathbb{F}} \to \mathrm{Set}^{\mathbb{F}}$$

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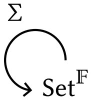

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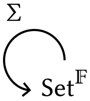

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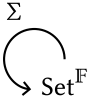

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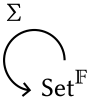

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Confluence of Term Rewriting Systems

β rewrite rule for λ-calculus:

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Confluence:

$$(\lambda x.(\lambda w.w)x)((\lambda y.y)z)$$

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$$z$$

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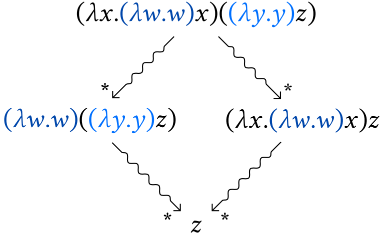

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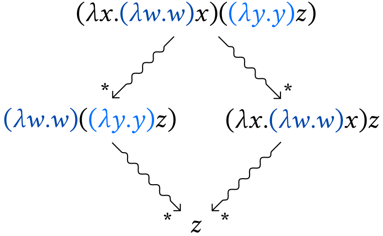
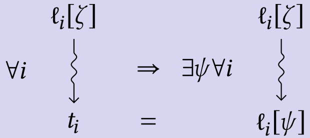


Coherence of $(\mathfrak{M}, \ell, r)$:

(1)



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Coherence of $(\mathfrak{M}, \ell, r)$:

(1) $$\forall i\ \ \ell_i[\zeta] \rightsquigarrow t_i \quad \Rightarrow \quad \exists \psi \forall i\ \ \ell_i[\zeta] \rightsquigarrow \ell_i[\psi], \quad t_i = \ell_i[\psi]$$

(2) $$\forall i\ \ \ell_i[\zeta] \rightsquigarrow \ell_i[\psi] \quad \Rightarrow \quad r[\zeta] \rightsquigarrow r[\psi]$$

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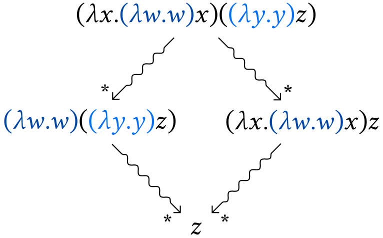

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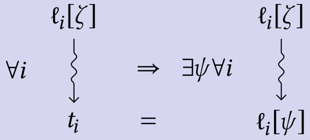

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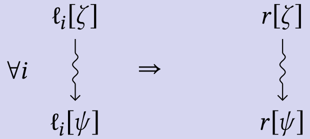

coherence of rewrite rule ⇒ confluence of reduction relation

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