Premonoidal and Kleisli double categories

Bojana Femić

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Mathematical Institute of
Serbian Academy of Sciences and Arts
Belgrade (Serbia)

Overview: two main parts

1. Premonoidality
   - double funny and quasi-functors,
   - classification of binoidal structures,
   - premonoidality vs monoidality

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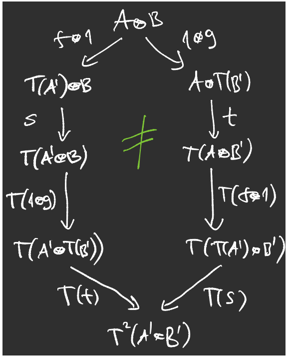
   - classification of binoidal structures,
   - premonoidality vs monoidality
2. Strong double monads and Kleisli double categories

Motivation from effectful languages

Effectful programming languages



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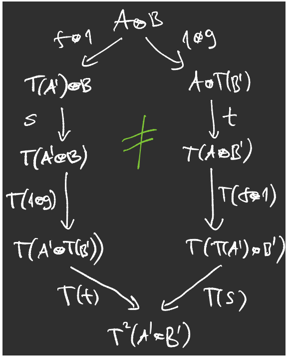

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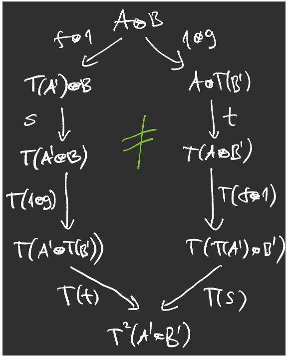

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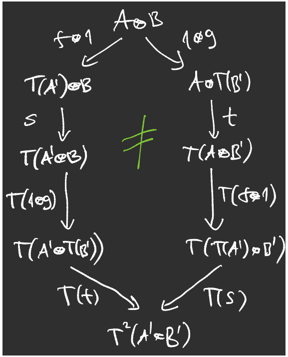

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Premonoidality

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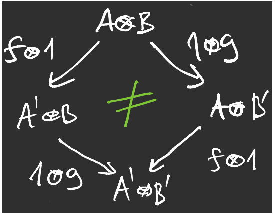

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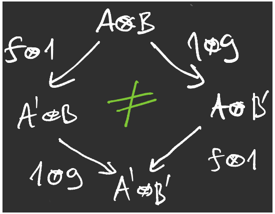

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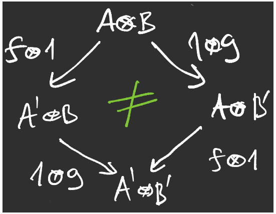
In all the settings (dimensions) a **premonoidal structure** encompasses:

- a binoidal structure $A \ltimes -,\ - \rtimes B$ (functors) ,
- $\alpha_{-,B,C},\ \alpha_{A,-,C},\ \alpha_{A,B,-},\quad \lambda_{-},\ \rho_{-}$ (tr.),
- 4 pentagons + 6 triangles (modif.) $(p_{-,-,-,-},\ m_{-,-},\ l_{-,-},\ r_{-,-})$.

Premonoidality and (tensor) products

The non-coincidence



is encoded by:

- funny functors (funny product)
- quasi-functors (Gray product).

Double funny and quasi-functors

Double funny- and quasi-functors

double functor $\Leftrightarrow$ double quasi-functor $H\colon \mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{C}$
$\mathcal{F}: \mathbb{A} \longrightarrow [\![\mathbb{B}, \mathbb{C}]\!]$

Double funny- and quasi-functors

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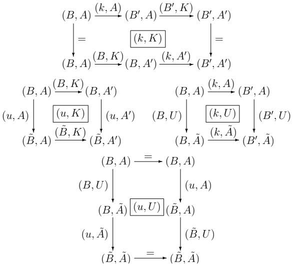
1. $(-,A)\colon \mathbb{B} \longrightarrow \mathbb{C}$ $\quad (B,-)\colon \mathbb{A} \longrightarrow \mathbb{C}$ double functors
2. 

3. satisfying 20/16/11/0 axioms

Funny and quasi multicategories

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$\rightsquigarrow$ double cats f_n-$[\mathbb{A}_1 \times \cdots \times \mathbb{A}_n, \mathbb{B}]$ and q_n-$[\mathbb{A}_1 \times \cdots \times \mathbb{A}_n, \mathbb{B}]$

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$\rightsquigarrow$ biclosed monoidal cats $(Dbl, \square_f)$, $(Dbl, \square)$, $(Dbl, \otimes)$

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Def. A strict premonoidal double cat. is a monoid in $(Dbl, \square_f)$.

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Def. A strict premonoidal double cat. is a monoid in $(Dbl, \square_f)$.

Thm. A pseudomonoid in the monoidal 2-cat. $(Dbl_2, -\square_2-)$ is a premonoidal double cat. (with a strict binoidal str. given by a funny functor and satisfying the $0 + 6 + 6 + 6$ axioms).

Classification of binoidal structures

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$(q_n\text{-}\,\mathbb{P}\mathrm{s}^{st}(\mathbb{D}^n, \mathbb{D}) \cong \mathbb{P}\mathrm{s}(\mathbb{D}^n, \mathbb{D}))$

Lifting of vertical structures

Lifting vertical structures and relation to bicategories

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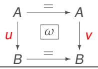
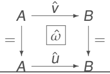
liftable vertical struct.	**lift to horizontal str.**	**yield in underlying $\mathcal{H}(\mathbb{D})$**
vert. tr. σ	horiz. tr. $\hat{\sigma}$	ps. nat. tr. $\mathcal{H}(\hat{\sigma})$
identity vert. modif. 	inv. horiz. modif. 	bicat. modif. clear

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monoidal d. cat. α, λ, ρ vert. tr. p, m, l, r identity vert. modif.	horizontally monoidal d. cat. $\hat{\alpha}, \hat{\lambda}, \hat{\rho}$ horiz. tr. $\hat{p}, \hat{m}, \hat{l}, \hat{r}$ inv. horiz. modif.	monoidal bicat. $\mathcal{H}(\hat{\alpha}), \mathcal{H}(\hat{\lambda}), \mathcal{H}(\hat{\rho})$ $\mathcal{H}(\hat{p}), \mathcal{H}(\hat{m}), \mathcal{H}(\hat{l}), \mathcal{H}(\hat{r})$

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premonoidal d. cat. $(\mathbb{D}, \alpha, \lambda, \rho)$	$\Rightarrow$	premonoidal bicategory $(\mathcal{H}(\mathbb{D}), \mathcal{H}(\hat{\alpha}), \mathcal{H}(\hat{\lambda}), \mathcal{H}(\hat{\rho}))$

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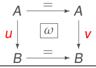
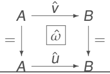
identity vert. modif. 	inv. horiz. modif. 	bicat. modif. clear
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$\mathbb{D}, \mathbb{Z}_p^{st}(\mathbb{D})$ monoidal	$\Rightarrow$	$\mathcal{H}(\mathbb{D}), \mathcal{Z}_p(\mathcal{H}(\mathbb{D}))$ monoidal

(Bi)strong double monads

(Bi)strong double monads and actions

notion	vertical		horizontal
	vert. tr. id. vert. modif.		horiz. tr. inv. horiz. modif.
double monads[known]	●		●
strength	●		●
action	●		●
bistrength	●	**mixed**	●

Strength vs extensions * bistrength yields premonoidality

Teo. (1v)

vertic. mon. $(\mathbb{D}, \alpha, \lambda, \rho)$ $\Rightarrow$ **vertic. act.** $\mathbb{D} \triangleright \mathbb{K}l(\hat{T})$ $\Rightarrow$ **horiz. act.** $\hat{\mathbb{D}} \triangleright \mathbb{K}l(\hat{T})$

vertic. (T, μ, η) — vertic. icon. θ — horiz. icon. $\hat{\theta}$

vertic. strength t

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extension of the canonical action $\mathbb{D} \triangleright \mathbb{D}$

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Teo. (1h)

horiz. mon. $(\mathbb{D}, \alpha, \lambda, \rho)$ $\Leftrightarrow$ **extension**

horiz. (S, μ, η)

horiz. strength s

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horiz. strength s

Teo. (2v)

vertic. mon. $(\mathbb{D}, \alpha, \lambda, \rho)$ $\Rightarrow$ $\mathbb{K}l(\hat{T})$ is premonoidal

vertic. (T, μ, η)

bistrength (t, s, q)

Strength vs extensions * bistrength yields premonoidality

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bistrength (t, s, q) $\rightsquigarrow$ $\alpha_{A,B,-}$, $\alpha_{-,B,C}$