# Monilmorphisms and relative extensivity

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Stellenbosch University

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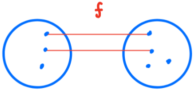

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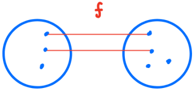


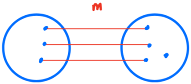

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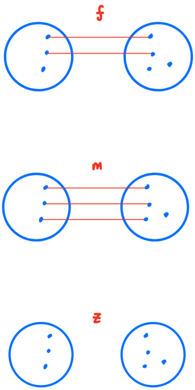

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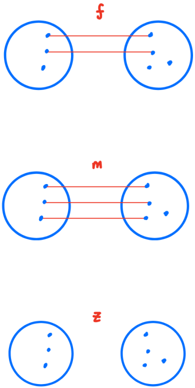


## In **Set**$^*$

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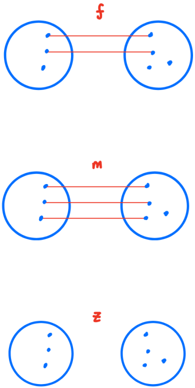


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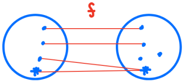

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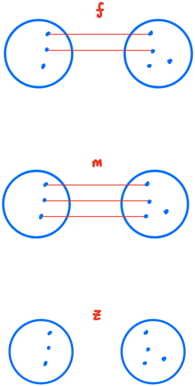


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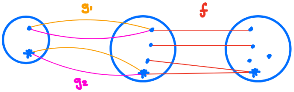

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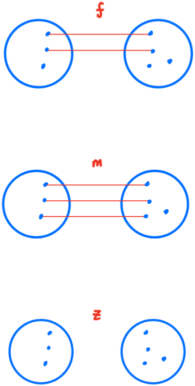


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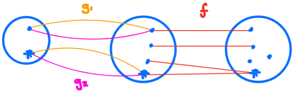


In any regular category:

$$\ker(f) \wedge \mathrm{im}(g_i) = 0$$
$$\iff$$
$$\ker(g_i) = \ker(fg_i)$$

# Monilmorphisms

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## Definition

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- Monilmorphisms are stable under pullback along monomorphisms.

# Monilmorphisms in categories of partial maps

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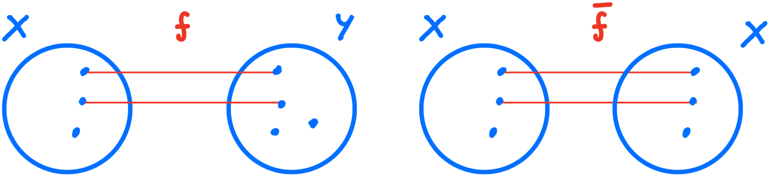
(R.1) $f\overline{f} = f$

(R.2) $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ whenever $\mathrm{dom}(f) = \mathrm{dom}(g)$

(R.3) $\overline{g\overline{f}} = \overline{g}\,\overline{f}$ whenever $\mathrm{dom}(f) = \mathrm{dom}(g)$

(R.4) $\overline{g}f = f\overline{gf}$ whenever $\mathrm{cod}(f) = \mathrm{dom}(g)$



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### Theorem

*In a restriction category with restriction zero T.F.A.E.*

1. *The monilmorphisms are exactly the restricted monics.*
2. *Every restriction idempotent is a monilmorphism.*

# Relative extensivity in $(1 \downarrow \mathbb{C})$

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$f$ is *extensive* [2] if extensivity condition holds for $f$

$$
\begin{array}{ccccc}
A_1 & \longrightarrow & A & \longleftarrow & A_2 \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} & \llcorner & \big\downarrow \\
X_1 & \longrightarrow & X_1 + X_2 & \longleftarrow & X_2
\end{array}
\quad \Longleftrightarrow \quad
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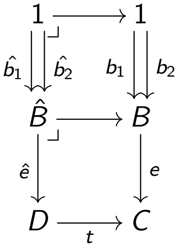
# Relative extensivity in $(1 \downarrow \mathbb{C})$

$f$ is *extensive* [2] if extensivity condition holds for $f$

$\mathbb{C}$ is PCTS if pulling the coequaliser of points back along $t$ with $\ker(t) = 0$ is coequaliser of points



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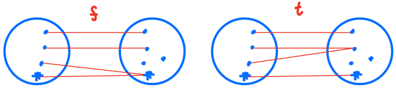
$\{f \mid f \text{ is monil}\} \underset{\sim}{\perp} \{t \mid \ker(t) = 0\}$



# Sum structures

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A *sum structure* [3] on $\mathbb{C}$ is a monoidal structure $\oplus$ with unit 0 and

$$\begin{array}{ccccc}
X_1 \oplus 0 & \xrightarrow{1 \oplus !} & X_1 \oplus X_2 & \xleftarrow{! \oplus 1} & 0 \oplus X_2 \\
\uparrow \rho & \nearrow \iota_1 & & \nwarrow \iota_2 & \uparrow \lambda \\
X_1 & & & & X_2
\end{array}$$

jointly epimorphic.

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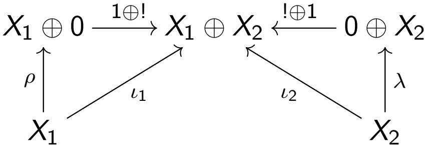


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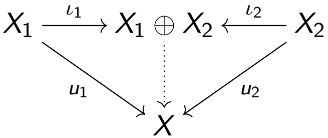
$u_1 \sqsubset u_2$



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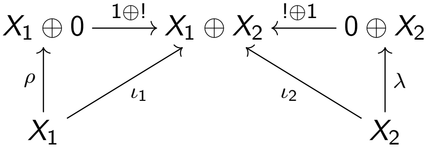


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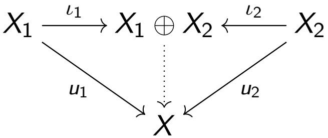


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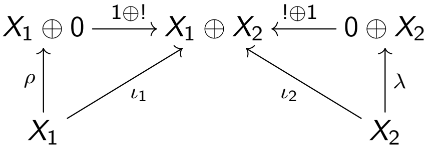


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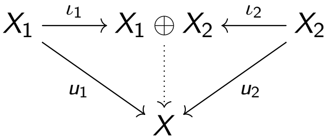


$f$ is $\sqsubset$-*reflecting* if $fu_1 \sqsubset fu_2 \implies u_1 \sqsubset u_2$

$f$ is extensive w.r.t. $\oplus$ if it satisfies extensivity condition with $+$ replaced with $\oplus$.

# Monilmorphisms and relative extensivity

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We will say $\mathbb{C}$ is PCSEP if the pullback of the square of a coequaliser of points is epi

$$1 \underset{b_2}{\overset{b_1}{\rightrightarrows}} B \overset{e}{\twoheadrightarrow} C$$

$$\begin{array}{ccc} P & \xrightarrow{\widehat{e \times e}} & A \\ \downarrow & & \downarrow g \\ B \times B & \xrightarrow[e \times e]{} & C \times C \end{array}$$

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## Theorem

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So in $(1 \downarrow \mathbb{C})_{\mathbf{Monil}}$ coproduct becomes sum structure $\oplus$

If $\mathbb{C}$ is also PCTS then in $(1 \downarrow \mathbb{C})_{\mathbf{Monil}}$

$$f \text{ extensive} \iff f \text{ mono} \iff f \sqsubset\text{-reflecting}$$

# Thank you!

Thank you for listening.

# References


[1] J.R.B. Cockett and S. Lack, *Restriction categories I: categories of partial maps*, Theoret. Comput. Sci 270 (2002), no. 1-2, 223–259.

[2] M. Hoefnagel and E. Theart, On extensivity and coextensivity of morphisms, Theory and Applications of Categories, 2025 (to appear).

[3] Z. Janelidze, Cover Relations on Categories, Applied Categorical Structures 17, 2009, 351–371.