A Categorical Framework for Generalized Compactness

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Core Question

Is there a single, underlying notion of compactness that unifies these concepts in a general categorical setting?

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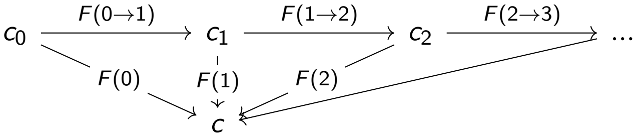

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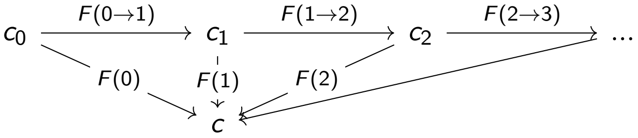

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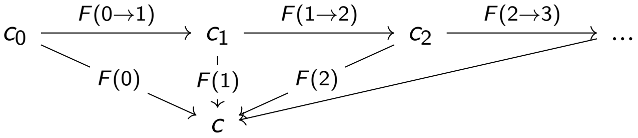

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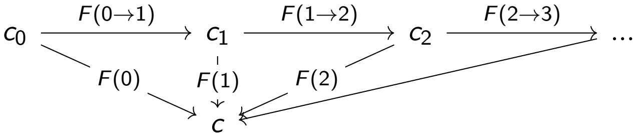

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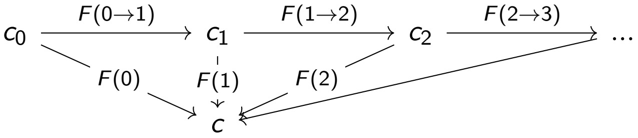

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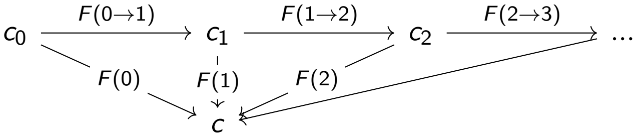

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  - $I = \mathcal{P}(X)$, for an X-indexed covers (Compactness).
- Stabilization means the process trivializes early at $i \in \mathrm{Obj}(I)$;
  - This object i must be from a pre-chosen class **designated small objects** and $F(k)$ must be an isomorphism for morphisms $i \xrightarrow{k} j$ in I.

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τ-Compactness

An object c is τ-**compact** if every covering $(F\colon I \to C/c, A, K, L) \in \tau_c$ of c **stabilizes**; there is a designated small object $i_0 \in A$ where $F(k)$ is an isomorphism for every $i_0 \to i \xrightarrow{k} j$ in I.

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- **Compact:** τ_c consists of covariant functors $F\colon \mathcal{P}(I) \to C/X$ induced from open covers $(U_i)_{i \in I}$ of the space X with designated small objects as finite subsets of I.

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Application: Protomodularity & Closure Properties

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Compactness is preserved under appropriate quotients

Let C be a category with pullbacks, a stable left-cancelable system M and a coverage τ subordinated to M[a]. Let $f\colon x \to y$ be stably M-extremal epimorphism in C. If x is τ-compact, then so is y.

[a] $F(i) \in M$ for $F\colon I \to C/c \in \tau_c$ and $i \in \mathrm{Obj}(I)$

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The Role of Protomodularity

Protomodularity provides the good behavior needed for stronger results. We use the tool of a **protomodular pre-factorization system**. Protomodularity ensures:

- Closure of τ-compact objects under **extensions**.
- Closure under **products** in pointed settings.
- The **Hopfian property** for Noetherian objects.

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3. **Extensions:** $a \to b \to c$ is a weak extension[a] problem with a, c τ-compact $\Rightarrow$ b is τ-compact.

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- **Abelian Categories:** The full subcategory of J-compact objects in an abelian category form a an exact subabelian category.

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Thank you!

Preprint soon available

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