Enrichment and families over a virtual double category

Soichiro Fujii
(Masaryk University)

Stephen Lack
(Macquarie University)

[Fujii and Lack, The familial nature of enrichment over virtual double categories, arXiv:2507.05529]

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$$\Longrightarrow \quad \begin{tikzcd} \mathrm{MonCAT}_{\mathrm{lax}} \arrow[r] & 2\text{-}\mathrm{CAT} \\ \mathcal{V} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \mathcal{V}\text{-}\mathrm{Cat} \arrow[u, phantom, sloped, "\in"] \end{tikzcd} \quad : \text{2-functor}$$

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$\boxed{\text{Q.}}$ What properties does Enr have?

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Enrichment over virtual double categories [Leinster 2002]

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$$\begin{tikzcd}
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k \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \begin{pmatrix} Rk \\ \downarrow R!_k \\ R1 \end{pmatrix} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

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\end{tikzcd}$$

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$$\begin{tikzcd}
& \mathcal{L}/R1 \arrow[dr, "\text{forgetful}"] \arrow[dl, dashed, bend right] & \\
\mathcal{K} \arrow[ur, "R_1"'] \arrow[ur, phantom, bend left, "\perp"] \arrow[rr, "R"'] & & \mathcal{L}
\end{tikzcd}$$

Enlarging the class of enrichment bases

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\{\text{monoidal categories}\} \arrow[r, hook] \arrow[d, hook] & \{\text{multicategories}\} \\
\{\text{bicategories}\} \arrow[d, hook] & \\
\{\text{pseudo double categories}\} &
\end{tikzcd}$$

[Betti, Carboni, Street, Walters, ...
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(Used by e.g.
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$$\begin{tikzcd} & 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1) \arrow[dl, shift right=2, "\mathbb{L}"'] \arrow[dr, "\text{forgetful}"] & \\ \mathrm{VDBL} \arrow[ur, shift right=2, "\mathrm{Enr}_1"'] \arrow[ur, phantom, "\perp"] \arrow[rr, "\mathrm{Enr}"'] & & 2\text{-}\mathrm{CAT} \end{tikzcd}$$

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- horizontal morphisms $$\begin{tikzcd} A \arrow[r, "T", "|" marking] & A' \end{tikzcd}$$, ...

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- horizontal morphisms $A \xrightarrow{T} A', \dots$
- multicells

$$\begin{tikzcd}
A_0 \arrow[r, "T_1", "|" marking] \arrow[d, "u"'] \arrow[drrr, phantom, "\alpha"] & A_0 \arrow[r, "T_2", "|" marking] & \cdots \arrow[r, "T_n", "|" marking] & A_n \arrow[d, "u'"] \\
B \arrow[rrr, "S"', "|" marking] & & & B'
\end{tikzcd}, \dots$$

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+ operations

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} vertical category $\mathbb{A}_{\mathrm{vert}}$

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$$\begin{tikzcd} A \arrow[r, "T", "\shortmid"{marking}] \arrow[d, equal] & A' \arrow[d, equal] \\ A \arrow[r, "T"', "\shortmid"{marking}] \arrow[ur, phantom, "\mathrm{id}_T"] & A' \end{tikzcd}$$

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$$\begin{tikzcd}
A_0 \arrow[r, "|" marking, "\Gamma_1"] \arrow[d, "u"'] & A_0 \arrow[r, "|" marking, "\Gamma_2"] \arrow[d, phantom, "\alpha"] & \cdots \arrow[r, "|" marking, "\Gamma_n"] & A_n \arrow[d, "u'"] \\
B \arrow[rrr, "|" marking, "\Delta"'] & & & B'
\end{tikzcd}$$

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$$\begin{tikzcd}
A \arrow[r, "|" marking, "\Gamma"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{id}_\Gamma"] & A' \arrow[d, equal] \\
A \arrow[r, "|" marking, "\Gamma"'] & A'
\end{tikzcd}$$

$$\begin{tikzcd}
& A_0 \arrow[rr, "|" marking, "\Gamma_1"] \arrow[dl, "u"'] \arrow[dr] \arrow[d, phantom, "\alpha_1"] & & A_1 \arrow[r, "|" marking, "\Gamma_2"] \arrow[d, phantom, "\alpha_2"] & A_2 \arrow[r, "|" marking, "\Gamma_3"] \arrow[d] & A_3 \arrow[d, "u'"] \arrow[dl, phantom, "\alpha_3"] \\
B_0 \arrow[rr, "|" marking] \arrow[d, "v"'] & \arrow[d, phantom, "\beta"] & B_1 \arrow[rr, "|" marking] & & B_2 \arrow[r, "|" marking] & B_3 \arrow[d, "v'"] \\
C \arrow[rrrrr, "|" marking, "\Theta"'] & & & & & C'
\end{tikzcd}
\longmapsto
\begin{tikzcd}
A_0 \arrow[r, "|" marking, "\Gamma_1"] \arrow[d, "vu"'] & A_1 \arrow[r, "|" marking, "\Gamma_2"] \arrow[d, phantom, "\beta\circ(\alpha_1{,}\alpha_2{,}\alpha_3)"] & A_2 \arrow[r, "|" marking, "\Gamma_3"] & A_3 \arrow[d, "v'u'"] \\
C \arrow[rrr, "|" marking, "\Theta"'] & & & C'
\end{tikzcd}$$

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An $\mathbb{A}$-category $\mathbb{C}$ consists of :

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$$\begin{tikzcd} \mathrm{ob}\,\mathbb{C} \arrow[r, "|-|^{\mathbb{C}}"] & \mathrm{ob}\,\mathbb{A} \\ x \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & {|x|^{\mathbb{C}}} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

The extent of x.

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$$\begin{tikzcd} \mathrm{ob}\, C \arrow[r, "|-|^{C}"] & \mathrm{ob}\, \mathbb{A} \\ x \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & {|x|^{C}} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

The extent of x.

- $\forall x, y \in C$, a horiz. mor.

$$|x|^{C} \xrightarrow{C(x,y)} |y|^{C} \text{ in } \mathbb{A}$$

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\end{tikzcd} \text{ in } \mathbb{A}$$

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$$\begin{tikzcd}[column sep=small]
& {|x|^C} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta^C_x"] & \\
{|x|^C} \arrow[rr, "{C(x,x)}"', "|" marking] & {} & {|x|^C}
\end{tikzcd}
\qquad
\begin{tikzcd}
{|x|^C} \arrow[r, "{C(x,y)}", "|" marking] \arrow[d, equal] & {|y|^C} \arrow[r, "{C(y,z)}", "|" marking] \arrow[d, phantom, "\mu^C_{xyz}"] & {|z|^C} \arrow[d, equal] \\
{|x|^C} \arrow[rr, "{C(x,z)}"', "|" marking] & {} & {|z|^C}
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satisfying the category axioms.

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- multicells

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An $\mathbb{A}$-functor $C \xrightarrow{F} D$ consists of:

Def. [Leinster 2002]

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An $\mathbb{A}$-category C consists of:

- a set $\mathrm{ob}\, C$ of objects
- a function $\mathrm{ob}\, C \xrightarrow{|-|^C} \mathrm{ob}\, \mathbb{A}$

$$\begin{tikzcd} \mathrm{ob}\, C \arrow[r, "|-|^C"] & \mathrm{ob}\, \mathbb{A} \\ x \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & {|x|^C} \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

The extent of x.

- $\forall x, y \in C$, a horiz. mor.

$$|x|^C \xrightarrow[\quad]{C(x,y)} |y|^C \text{ in } \mathbb{A}$$

- multicells

$$\begin{tikzcd} & {|x|^C} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta^C_x"] & \\ {|x|^C} \arrow[rr, "|" marking, "C(x,x)"'] & {} & {|x|^C} \end{tikzcd} \qquad \begin{tikzcd} {|x|^C} \arrow[r, "|" marking, "C(x,y)"] \arrow[d, equal] & {|y|^C} \arrow[r, "|" marking, "C(y,z)"] \arrow[d, phantom, "\mu^C_{xyz}"] & {|z|^C} \arrow[d, equal] \\ {|x|^C} \arrow[rr, "|" marking, "C(x,z)"'] & {} & {|z|^C} \end{tikzcd} \quad \text{in } \mathbb{A}$$

satisfying the category axioms.

An $\mathbb{A}$-functor $C \xrightarrow{F} D$ consists of:

- a function

$$\begin{tikzcd} \mathrm{ob}\, C \arrow[r, "\mathrm{ob} F"] & \mathrm{ob}\, D \\ x \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & Fx \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

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An $\mathbb{A}$-category $\mathcal{C}$ consists of :

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The extent of x.

- $\forall x, y \in \mathcal{C}$, a horiz. mor.

$$\begin{tikzcd} |x|^{\mathcal{C}} \arrow[r, "\mathcal{C}(x,y)", "\shortmid"{marking}] & |y|^{\mathcal{C}} \end{tikzcd} \text{ in } \mathbb{A}$$

- multicells

$$\begin{tikzcd}[column sep=small] & |x|^{\mathcal{C}} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta^{\mathcal{C}}_x"] & \\ |x|^{\mathcal{C}} \arrow[rr, "\mathcal{C}(x,x)"', "\shortmid"{marking}] & {} & |x|^{\mathcal{C}} \end{tikzcd} \qquad \begin{tikzcd} |x|^{\mathcal{C}} \arrow[r, "\mathcal{C}(x,y)", "\shortmid"{marking}] \arrow[d, equal] & |y|^{\mathcal{C}} \arrow[r, "\mathcal{C}(y,z)", "\shortmid"{marking}] \arrow[d, phantom, "\mu^{\mathcal{C}}_{xyz}"] & |z|^{\mathcal{C}} \arrow[d, equal] \\ |x|^{\mathcal{C}} \arrow[rr, "\mathcal{C}(x,z)"', "\shortmid"{marking}] & {} & |z|^{\mathcal{C}} \end{tikzcd} \quad \text{in } \mathbb{A}$$

satisfying the category axioms.

An $\mathbb{A}$-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ consists of:

- a function

$$\begin{tikzcd} \mathrm{ob}\,\mathcal{C} \arrow[r, "\mathrm{ob}F"] & \mathrm{ob}\,\mathcal{D} \\ x \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & Fx \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

- $\forall x \in \mathcal{C}$, a vertical mor.

$$\begin{tikzcd} |x|^{\mathcal{C}} \arrow[d, "F_x"] \\ |Fx|^{\mathcal{D}} \end{tikzcd} \text{ in } \mathbb{A}$$

Def. [Leinster 2002]

$\mathbb{A}$: virtual double category

An $\mathbb{A}$-category C consists of :

- a set $\mathrm{ob}\,C$ of objects
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The extent of x.

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$$\begin{tikzcd} {|x|^C} \arrow[r, "{C(x,y)}", "\shortmid"{marking}] & {|y|^C} \end{tikzcd} \text{ in } \mathbb{A}$$

- multicells

$$\begin{tikzcd} & {|x|^C} \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta^C_x"] & \\ {|x|^C} \arrow[rr, "{C(x,x)}"', "\shortmid"{marking}] & {} & {|x|^C} \end{tikzcd} \qquad \begin{tikzcd} {|x|^C} \arrow[r, "{C(x,y)}", "\shortmid"{marking}] \arrow[d, equal] & {|y|^C} \arrow[r, "{C(y,z)}", "\shortmid"{marking}] \arrow[d, phantom, "\mu^C_{xyz}"] & {|z|^C} \arrow[d, equal] \\ {|x|^C} \arrow[rr, "{C(x,z)}"', "\shortmid"{marking}] & {} & {|z|^C} \end{tikzcd} \text{ in } \mathbb{A}$$

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$$\begin{tikzcd}[column sep=small] & {|x|^C} \arrow[dl, equal] \arrow[dr, equal] & \\ {|x|^C} \arrow[rr, "{C(x,x)}"', "\shortmid"{marking}] \arrow[rr, phantom, bend left=30, "\eta^C_x"] & & {|x|^C} \end{tikzcd} \qquad \begin{tikzcd} {|x|^C} \arrow[r, "{C(x,y)}", "\shortmid"{marking}] \arrow[d, equal] & {|y|^C} \arrow[r, "{C(y,z)}", "\shortmid"{marking}] \arrow[d, phantom, "\mu^C_{xyz}"] & {|z|^C} \arrow[d, equal] \\ {|x|^C} \arrow[rr, "{C(x,z)}"', "\shortmid"{marking}] & & {|z|^C} \end{tikzcd} \quad \text{in } \mathbb{A}$$

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$\Longrightarrow$ $\mathbb{A}$-Cat : 2-category

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satisfying the functor axioms.

$\Longrightarrow$ $\mathbb{A}$-Cat : 2-category

$$\begin{tikzcd} \mathbf{VDBL} \arrow[r, "\mathrm{Enr}"] & \text{2-}\mathbf{CAT} \\ \mathbb{A} \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & \mathbb{A}\text{-}\mathbf{Cat} \arrow[u, phantom, "\in" sloped] \end{tikzcd} : \text{2-functor}$$

$$\begin{tikzcd}\mathrm{VDBL} \arrow[r, leftarrow, shift left=2, "\mathbb{L}"] \arrow[r, shift right=2, "\mathrm{Enr}_1"'] \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1)\end{tikzcd}$$

What is $\mathbb{V}\mathrm{DBL} \underset{\mathrm{Enr}_1}{\overset{\mathbb{L}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1)$?

What is
$$\begin{tikzcd} \mathrm{VDBL} \arrow[r, shift right=1.5, "\mathrm{Enr}_1"'] \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1) \arrow[l, shift right=1.5, "\mathbb{L}"'] \end{tikzcd}$$
?

$\mathrm{Enr}(1) =$ "1-Cat"

What is $\begin{tikzcd} \mathrm{VDBL} \arrow[r, shift right, "\mathrm{Enr}_1"'] \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1) \arrow[l, shift right, "\mathbb{L}"'] \end{tikzcd}$?

$$\begin{tikzcd} S \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\exists!"] \\ T \end{tikzcd}$$

$\mathrm{Enr}(1) =$ "1-Cat" $= \mathrm{Set}_{lc}$: locally chaotic 2-category whose underlying category is Set.

What is $\mathrm{VDBL} \underset{\mathrm{Enr}_1}{\overset{\mathbb{L}}{\underset{\perp}{\leftrightarrows}}} \text{2-CAT}/\mathrm{Enr}(1)$?

$$\begin{tikzcd} S \arrow[d, bend right=60, "f"'] \arrow[d, bend left=60, "g"] \arrow[d, phantom, "\overset{\exists!}{\Rightarrow}"] \\ T \end{tikzcd}$$

$\mathrm{Enr}(1) =$ "1-Cat" $= \mathrm{Set}_{lc}$: locally chaotic 2-category whose underlying category is Set.

$$\mathrm{Enr}_1(\mathbb{A}) = \left(\begin{tikzcd} \mathrm{Enr}(\mathbb{A}) \arrow[d, "\mathrm{Enr}(!_{\mathbb{A}})"] \\ \mathrm{Enr}(1) \end{tikzcd} \right)$$

What is $\mathrm{VDBL} \underset{\mathrm{Enr}_1}{\overset{\mathbb{L}}{\underset{\perp}{\rightleftarrows}}} 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1)$?

$$\begin{tikzcd} S \arrow[d, bend right=50, "f"', ""{name=F}] \arrow[d, bend left=50, "g", ""'{name=G}] \\ T \arrow[Rightarrow, dotted, from=F, to=G, "\exists!"] \end{tikzcd}$$

$\mathrm{Enr}(1) =$ "$1\text{-}\mathrm{Cat}$" $= \mathrm{Set}_{lc}$: locally chaotic 2-category whose underlying category is Set.

$$\mathrm{Enr}_1(\mathbb{A}) = \begin{pmatrix} \mathrm{Enr}(\mathbb{A}) \\ \big\downarrow \mathrm{Enr}(!_{\mathbb{A}}) \\ \mathrm{Enr}(1) \end{pmatrix} = \begin{pmatrix} \mathbb{A}\text{-}\mathrm{Cat} \\ \big\downarrow \mathrm{ob} \\ \mathrm{Set}_{lc} \end{pmatrix}.$$

What is
$$\begin{tikzcd} \mathrm{VDBL} \arrow[r, shift right=1.5, "\mathrm{Enr}_1"'] \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1) \arrow[l, shift right=1.5, "\mathbb{L}"'] \end{tikzcd}$$
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Given $\left(\begin{tikzcd} \mathbb{K} \arrow[d, "G"] \\ \mathrm{Set}_{lc} \end{tikzcd}\right) \in 2\text{-}\mathrm{CAT}/\mathrm{Set}_{lc}$, $\mathbb{L}G \in \mathrm{VDBL}$ is . . .

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$$\left(\begin{tikzcd} \mathcal{K}_0 \arrow[d, "\hat{G}"] \\ \mathbf{Set} \end{tikzcd}\right) \text{ by } 2\text{-}\mathbf{CAT} \underset{(-)_{lc}}{\overset{(-)_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CAT}$$

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- $\underline{\text{obj}}$. $(K \in \mathbb{K}, x \in GK)$

What is $$\begin{tikzcd} \mathrm{VDBL} \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1) \arrow[l, bend right, "\mathbb{L}"'] \arrow[from=1-1, to=1-2, bend right, "\mathrm{Enr}_1"'] \end{tikzcd}$$?

$$\begin{tikzcd} S \arrow[d, bend right=60, "f"'] \arrow[d, bend left=60, "g"] \arrow[d, phantom, "\exists!"] \\ T \end{tikzcd}$$

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$$\begin{pmatrix} \mathbb{K}_0 \\ \big\downarrow \hat{G} \\ \mathrm{Set} \end{pmatrix}$$ by $$\begin{tikzcd} 2\text{-}\mathrm{CAT} \arrow[r, bend left, "(-)_0"] \arrow[r, phantom, "\perp"] & \mathrm{CAT} \arrow[l, bend left, "(-)_{lc}"] \end{tikzcd}$$

Given $\begin{pmatrix} \mathbb{K} \\ \big\downarrow G \\ \mathrm{Set}_{lc} \end{pmatrix} \in 2\text{-}\mathrm{CAT}/\mathrm{Set}_{lc}$, $\mathbb{L}G \in \mathrm{VDBL}$ is ...

- <u>obj.</u> $(K \in \mathbb{K}, x \in GK)$
- <u>v.mor.</u> $\begin{matrix} (K,x) \\ \big\downarrow \\ (L,y) \end{matrix}$ in $\mathbb{L}G$ = $\begin{matrix} K \\ \big\downarrow f \\ L \end{matrix}$ in $\mathbb{K}$ s.t. $(Gf)x = y$.

What is $\mathrm{VDBL} \underset{\mathrm{Enr}_1}{\overset{\mathbb{L}}{\underset{\perp}{\leftrightarrows}}} 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1)$?

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$\}$ $(\mathbb{L}G)_{\mathrm{vert}} = \mathrm{El}(\hat{G})$

What is $\mathbf{VDBL} \underset{\mathrm{Enr}_1}{\overset{\mathbb{L}}{\underset{\perp}{\leftrightarrows}}} 2\text{-}\mathbf{CAT}/\mathrm{Enr}(1)$?

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$$\left(\begin{tikzcd} \mathcal{K}_0 \arrow[d, "\hat{G}"] \\ \mathbf{Set} \end{tikzcd}\right) \text{ by } 2\text{-}\mathbf{CAT} \underset{(-)_{lc}}{\overset{(-)_0}{\underset{\perp}{\rightleftarrows}}} \mathbf{CAT}$$

Given $\left(\begin{tikzcd} \mathcal{K} \arrow[d, "G"] \\ \mathbf{Set}_{lc} \end{tikzcd}\right) \in 2\text{-}\mathbf{CAT}/\mathbf{Set}_{lc}$, $\mathbb{L}G \in \mathbf{VDBL}$ is...

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- <u>horiz. mor</u>

$\{(k,x) \nrightarrow (k',x')\} =$

What is $$\begin{tikzcd} \mathbb{VDBL} \arrow[r, bend right, "\mathrm{Enr}_1"'] \arrow[r, phantom, "\perp"] & \text{2-CAT}/\mathrm{Enr}(1) \arrow[l, bend right, "\mathbb{L}"'] \end{tikzcd}$$?

$$\begin{tikzcd} S \arrow[d, bend right=60, "f"'] \arrow[d, bend left=60, "g"] \arrow[d, phantom, "\overset{\exists!}{\Rightarrow}"] \\ T \end{tikzcd}$$

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Given $\left(\begin{tikzcd} \mathcal{K} \arrow[d, "G"] \\ \mathrm{Set}_{lc} \end{tikzcd}\right) \in \text{2-CAT}/\mathrm{Set}_{lc}$, $\mathbb{L}G \in \mathbb{VDBL}$ is...

- obj. $(K \in \mathcal{K}, x \in GK)$
- v.mor. $$\begin{tikzcd} (K,x) \arrow[d] \\ (L,y) \end{tikzcd} \text{ in } \mathbb{L}G = \begin{tikzcd} K \arrow[d, "f"] \\ L \end{tikzcd} \text{ in } \mathcal{K} \text{ s.t. } (Gf)x = y.$$

$\Big\}$ $(\mathbb{L}G)_{vert} = \mathrm{El}(\hat{G})$

- horiz.mor

$$\{(K,x) \nrightarrow (K',x')\} = \begin{cases} \{*\} & \text{if } K = K' \\ & \end{cases}$$

What is $$\begin{tikzcd} \mathbb{VDBL} \arrow[r, shift right, "\mathrm{Enr}_1"'] \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{CAT}/\mathrm{Enr}(1) \arrow[l, shift right, "\mathbb{L}"'] \end{tikzcd}$$?

$$\begin{tikzcd} S \arrow[d, bend right=50, "f"', ""{name=F}] \arrow[d, bend left=50, "g", ""'{name=G}] \\ T \arrow[from=F, to=G, Rightarrow, dotted, "\exists!"] \end{tikzcd}$$

$\mathrm{Enr}(1) =$ "$1\text{-}\mathrm{Cat}$" $= \mathrm{Set}_{lc}$: locally chaotic 2-category whose underlying category is Set.

$$\mathrm{Enr}_1(\mathbb{A}) = \begin{pmatrix} \mathrm{Enr}(\mathbb{A}) \\ \big\downarrow \mathrm{Enr}(!_{\mathbb{A}}) \\ \mathrm{Enr}(1) \end{pmatrix} = \begin{pmatrix} \mathbb{A}\text{-}\mathrm{Cat} \\ \big\downarrow \mathrm{ob} \\ \mathrm{Set}_{lc} \end{pmatrix}.$$

$$\begin{pmatrix} \mathbb{K}_0 \\ \big\downarrow \hat{G} \\ \mathrm{Set} \end{pmatrix} \text{ by } \begin{tikzcd} 2\text{-}\mathrm{CAT} \arrow[r, shift left, "(-)_0"] \arrow[r, phantom, "\perp"] & \mathrm{CAT} \arrow[l, shift left, "(-)_{lc}"] \end{tikzcd}$$

Given $\begin{pmatrix} \mathbb{K} \\ \big\downarrow G \\ \mathrm{Set}_{lc} \end{pmatrix} \in 2\text{-}\mathrm{CAT}/\mathrm{Set}_{lc}$, $\mathbb{L}G \in \mathbb{VDBL}$ is...

- obj. $(k \in \mathbb{K}, x \in Gk)$
- v. mor. $\begin{matrix} (k,x) \\ \big\downarrow \\ (l,y) \end{matrix}$ in $\mathbb{L}G$ = $\begin{matrix} k \\ \big\downarrow f \\ l \end{matrix}$ in $\mathbb{K}$ s.t. $(Gf)x = y$.

$(\mathbb{L}G)_{vert} = \mathrm{El}(\hat{G})$

- horiz. mor

$$\{(k,x) \nrightarrow (k',x')\} = \begin{cases} \{*\} & \text{if } k = k' \\ \emptyset & \text{if } k \neq k'. \end{cases}$$

Abstract proof of $\mathrm{VDBL} \xrightarrow{\mathrm{Enr}} \text{2-CAT}$: param. r. 2-adj. :

Abstract proof of VDBL $\xrightarrow{\text{Enr}}$ 2-CAT : param. r. 2-adj. :

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Observation

{param. right 2-adj.} is closed under composition and contains {right 2-adj.} and {polynomial 2-functors}.

Abstract proof of VDBL $\xrightarrow{\mathrm{Enr}}$ 2-CAT : param. r. 2-adj.:

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> $\{$param. right 2-adj.$\}$ is closed under composition and contains $\{$right 2-adj.$\}$ and $\{$polynomial 2-functors$\}$.

Enr is the composite

$$\mathrm{VDBL} \xrightarrow{\mathrm{Mat}} \mathrm{VDBL} \xrightarrow{\mathrm{Mod}} \mathrm{UVDBL} \xrightarrow{\mathcal{V}} \text{2-CAT}$$

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2-category of unital v.dbl. cats.

Abstract proof of VDBL $\xrightarrow{\mathrm{Enr}}$ 2-CAT : param. r. 2-adj.:

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$$\begin{tikzcd}
\mathrm{VDBL} \arrow[r, "\mathrm{Mat}"] \arrow[rr, bend right, "\mathrm{Enr}"'] & \mathrm{VDBL} \arrow[r, "\mathrm{Mod}"] & \mathrm{UVDBL} \arrow[r, "\mathcal{V}"] & \text{2-CAT} \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\
\mathbb{A} \arrow[rr, mapsto] & & \mathbb{A}\text{-}\mathbb{P}\mathrm{rof} &
\end{tikzcd}$$

2-category of unital v.dbl. cats.

$$\begin{tikzcd}
\mathrm{VDBL} \arrow[r, "\mathrm{Mat}"] & \mathrm{VDBL} \\
A \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & A\text{-}\mathrm{Mat} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

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\mathrm{VDBL} \arrow[r, "\mathrm{Mat}"] & \mathrm{VDBL} \\
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$\underline{\text{obj.}}$ $(I : \text{set}, (A_i \in A)_{i \in I})$

$$\begin{tikzcd} \mathbb{VDBL} \arrow[r, "\mathbb{M}\mathrm{at}"] & \mathbb{VDBL} \\ \mathbb{A} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \mathbb{A}\text{-}\mathbb{M}\mathrm{at} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

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v.mor.
$$\begin{tikzcd} (I, (A_i)_i) \arrow[d] \\ (J, (B_j)_j) \end{tikzcd} \quad \text{is} \quad \begin{tikzcd} I \arrow[d, "f"] \\ J \end{tikzcd}, \left(\begin{tikzcd} A_i \arrow[d, "u_i"] \\ B_{fi} \end{tikzcd} \text{ in } \mathbb{A} \right)_{i \in I}$$

$$\begin{tikzcd} \mathrm{VDBL} \arrow[r, "\mathrm{Mat}"] & \mathrm{VDBL} \\ \mathbb{A} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \mathbb{A}\text{-}\mathrm{Mat} \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

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$$\Bigg\} \; (\mathbb{A}\text{-}\mathrm{Mat})_{\mathrm{vert}} = \mathrm{Fam}(\mathbb{A}_{\mathrm{vert}}).$$

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$$\begin{tikzcd}
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\end{tikzcd}
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\end{tikzcd}
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\end{tikzcd}
\text{ in } \mathbb{A}
\right)_{i\in I}$$

$$\left.\begin{array}{l} \text{obj. and v.mor.} \end{array}\right\} (\mathbb{A}\text{-}\mathbb{M}\mathrm{at})_{\mathrm{vert}} = \mathrm{Fam}(\mathbb{A}_{\mathrm{vert}}).$$

<u>h.mor.</u>
$$\left(A_i \xrightarrow[\shortmid]{T_{ij}} B_j \text{ in } \mathbb{A}\right)_{i\in I, j\in J}$$
$$\begin{tikzcd}
(I, (A_i)_i) \arrow[r, "\shortmid" marking] & (J, (B_j)_j)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{VDBL} \arrow[r, "\mathrm{Mat}"] & \mathrm{VDBL} \\
\mathbb{A} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \mathbb{A}\text{-}\mathrm{Mat} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$\left.\begin{array}{ll}
\underline{\text{obj.}} & \left(I : \mathrm{Set}, \left(A_i \in \mathbb{A}\right)_{i\in I}\right) \\[1em]
\underline{\text{v.mor.}} & \begin{tikzcd} (I,(A_i)_i) \arrow[d] \\ (J,(B_j)_j) \end{tikzcd} \quad \text{is} \quad \begin{tikzcd} I \arrow[d, "f"] \\ J \end{tikzcd} , \left(\begin{tikzcd} A_i \arrow[d, "u_i"] \\ B_{fi} \end{tikzcd} \text{ in } \mathbb{A} \right)_{i\in I}
\end{array}\right\} (\mathbb{A}\text{-}\mathrm{Mat})_{\mathrm{vert}} = \mathrm{Fam}(\mathbb{A}_{\mathrm{vert}}).$$

$$\underline{\text{h.mor.}} \qquad \begin{tikzcd}[column sep=huge] (I,(A_i)_i) \arrow[r, "|", "{\left(A_i \xrightarrow{T_{ij}} B_j \text{ in } \mathbb{A}\right)_{i\in I, j\in J}}"'] & (J,(B_j)_j) \end{tikzcd}$$

is the composite

$$\begin{tikzcd}
\mathrm{VDBL} \arrow[r, "(\mathrm{Set}_*)_{hc}\times(-)"] & \mathrm{VDBL}/(\mathrm{Set}_*)_{hc} \arrow[r, "\Pi_p"] & \mathrm{VDBL}/\mathrm{Set}_{hc} \arrow[r, "\text{forgetful}"] & \mathrm{VDBL}
\end{tikzcd}$$

$$\begin{tikzcd} \mathbb{VDBL} \arrow[r, "\mathrm{Mat}"] & \mathbb{VDBL} \\ \mathbb{A} \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \mathbb{A}\text{-}\mathrm{Mat} \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

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h.mor.
$$\begin{tikzcd} (I, (A_i)_i) \arrow[rr, "|" marking, "{\left(A_i \xrightarrow{T_{ij}} B_j \text{ in } \mathbb{A}\right)_{i \in I, j \in J}}"] & & (J, (B_j)_j) \end{tikzcd}$$

is the composite

$$\begin{tikzcd} \mathbb{VDBL} \arrow[r, "(\mathrm{Set}_*)_{hc} \times (-)"] & \mathbb{VDBL}/(\mathrm{Set}_*)_{hc} \arrow[r, "\Pi_p"] & \mathbb{VDBL}/\mathrm{Set}_{hc} \arrow[r, "\text{forgetful}"] & \mathbb{VDBL} \end{tikzcd}$$

induced by

$$\begin{tikzcd} 1 & (\mathrm{Set}_*)_{hc} \arrow[l, "!"'] \arrow[r, "p"] & \mathrm{Set}_{hc} \arrow[r, "!"] & 1 \end{tikzcd}$$

in $\mathbb{VDBL}$.

$$\begin{tikzcd}
\mathrm{VDBL} \arrow[r, "\mathrm{Mat}"] & \mathrm{VDBL} \\
\mathbb{A} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & \mathbb{A}\text{-}\mathrm{Mat} \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

obj. $(I : \mathrm{Set}, (A_i \in \mathbb{A})_{i \in I})$

v.mor. $(I,(A_i)_i) \downarrow (J,(B_j)_j)$ is $I \overset{f}{\downarrow} J$, $\left(A_i \overset{u_i}{\downarrow} B_{fi} \text{ in } \mathbb{A} \right)_{i \in I}$

$\Big\}$ $(\mathbb{A}\text{-}\mathrm{Mat})_{\mathrm{vert}} = \mathrm{Fam}(\mathbb{A}_{\mathrm{vert}})$.

h.mor. $(A_i \xrightarrow[\mid]{T_{ij}} B_j \text{ in } \mathbb{A})_{i \in I, j \in J}$

$$(I,(A_i)_i) \xrightarrow{\quad \mid \quad} (J,(B_j)_j)$$

is the composite

$$\mathrm{VDBL} \xrightarrow{(\mathrm{Set}_*)_{hc} \times (-)} \mathrm{VDBL}/(\mathrm{Set}_*)_{hc} \xrightarrow{\Pi_P} \mathrm{VDBL}/\mathrm{Set}_{hc} \xrightarrow{\text{forgetful}} \mathrm{VDBL}$$

induced by

discrete opfibration

$\Rightarrow$ powerful (exponentiable) cf. [Bowler, 2011]

$$1 \xleftarrow{\ !\ } (\mathrm{Set}_*)_{hc} \xrightarrow{P} \mathrm{Set}_{hc} \xrightarrow{\ !\ } 1$$

in VDBL.

The discrete opfibration $(\mathrm{Set}_*)_{hc} \xrightarrow{P} \mathrm{Set}_{hc}$ is induced by $\mathrm{Set}_* \xrightarrow{\text{forgetful}} \mathrm{Set}$ via

$$\begin{tikzcd} \mathrm{CAT} \arrow[r, shift right, "(-)_{hc}"'] \arrow[r, phantom, "\perp"] & \mathrm{VDBL} \arrow[l, shift right, "(-)_{vert}"'] \end{tikzcd} .$$

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Now:

- $\mathrm{Set}_* \xrightarrow{\text{forgetful}} \mathrm{Set}$ is a universal disc. opfib. (w/ small fibers) in CAT.

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Now:

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The discrete opfibration $(\mathrm{Set}_*)_{hc} \xrightarrow{P} \mathrm{Set}_{hc}$ is induced by $\mathrm{Set}_* \xrightarrow{\text{forgetful}} \mathrm{Set}$ via

$$\begin{tikzcd}
\mathrm{CAT} \arrow[r, shift right=1.5, "(-)_{hc}"'] \arrow[r, phantom, "\perp"] & \mathrm{VDBL} \arrow[l, shift right=1.5, "(-)_{vert}"']
\end{tikzcd}.$$

Now:

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$$\begin{tikzcd}
& \mathrm{CAT}/\mathrm{Fam}(1) = \mathrm{CAT}/\mathrm{Set} \arrow[dl, shift right=1.5, "\mathrm{El}"'] \arrow[dr, "\text{forgetful}"] & \\
\mathrm{CAT} \arrow[ur, shift right=1.5, "\mathrm{Fam}_1"'] \arrow[ur, phantom, "\perp"] \arrow[rr, "\mathrm{Fam}"'] & & \mathrm{CAT}
\end{tikzcd}$$

(See e.g. [Johnson 1987])

The discrete opfibration $(\mathrm{Set}_*)_{hc} \xrightarrow{P} \mathrm{Set}_{hc}$ is induced by

$\mathrm{Set}_* \xrightarrow{\text{forgetful}} \mathrm{Set}$ via

$$\begin{tikzcd}
\mathrm{CAT} \arrow[r, shift right, "(-)_{hc}"'] \arrow[r, phantom, "\perp"] & \mathrm{VDBL} \arrow[l, shift right, "(-)_{vert}"']
\end{tikzcd}$$

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Now:

- $\mathrm{Set}_* \xrightarrow{\text{forgetful}} \mathrm{Set}$ is a universal disc. opfib. (w/ small fibers) in CAT.
- $1 \xleftarrow{!} \mathrm{Set}_* \xrightarrow{\text{forgetful}} \mathrm{Set} \xrightarrow{!} 1$ induces $\mathrm{CAT} \xrightarrow{\mathrm{Fam}} \mathrm{CAT}$, with

$$\begin{tikzcd}
\mathrm{El}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_* \arrow[d] \\
\mathcal{C} \arrow[r, "F"'] & \mathrm{Set}
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{CAT}/\mathrm{Fam}(1) = \mathrm{CAT}/\mathrm{Set} \arrow[dl, shift right, "\mathrm{El}"'] \arrow[dr, "\text{forgetful}"] & \\
\mathrm{CAT} \arrow[ur, shift right, "\mathrm{Fam}_1"'] \arrow[ur, phantom, "\perp"] \arrow[rr, "\mathrm{Fam}"'] & & \mathrm{CAT}
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  $$\mathrm{Span}_* \xrightarrow{Q} \mathrm{Span}$$

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<u>not</u> $(\mathbb{S}\mathrm{et}_*)_{hc} \xrightarrow{P} \mathbb{S}\mathrm{et}_{hc}$ (for $\mathbb{M}\mathrm{at}$), but

$$\begin{tikzcd} \mathbb{S}\mathrm{pan}_* \arrow[r, "Q"] \arrow[d, phantom, "\shortparallel"] & \mathbb{S}\mathrm{pan} \arrow[d, phantom, "\shortparallel"] \\ \mathbb{S}\mathrm{pan}(\mathrm{Set}_*) & \mathbb{S}\mathrm{pan}(\mathrm{Set}) \end{tikzcd}$$

$\mathcal{C}$: cat. w/ pb.

$\Rightarrow \mathbb{S}\mathrm{pan}(\mathcal{C})$: v. dbl. cat.

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\mathbb{S}\mathrm{pan}_* \arrow[r, "Q"] \arrow[d, equal] & \mathbb{S}\mathrm{pan} \arrow[d, equal] \\
\mathbb{S}\mathrm{pan}(\mathrm{Set}_*) & \mathbb{S}\mathrm{pan}(\mathrm{Set})
\end{tikzcd}$$

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\mathbb{S}\mathrm{pan}_* \arrow[r, "Q"] \arrow[d, equal] & \mathbb{S}\mathrm{pan} \arrow[d, equal] \\
\mathbb{S}\mathrm{pan}(\mathsf{Set}_*) & \mathbb{S}\mathrm{pan}(\mathsf{Set})
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Generalizes → [Paré 2011].

$$\begin{tikzcd}
 & \mathsf{VDBL}/\mathbb{S}\mathrm{pan} \arrow[dl, shift right, "\mathbb{E}\mathrm{lt}"'] \arrow[dr, "\text{forgetful}"] & \\
\mathsf{VDBL} \arrow[ur, shift right, "\mathbb{F}\mathrm{am}_1"'] \arrow[rr, "\mathbb{F}\mathrm{am}"'] \arrow[ur, phantom, "\dashv"] & & \mathsf{VDBL}
\end{tikzcd}$$