

# Enrichment and families over a virtual double category

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[Fujii and Lack, The familial nature of enrichment  
over virtual double categories, [arXiv:2507.05529](https://arxiv.org/abs/2507.05529)]

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Q. What properties does **Enr** have?

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Enrichment over  
virtual double categories [Leinster 2002]

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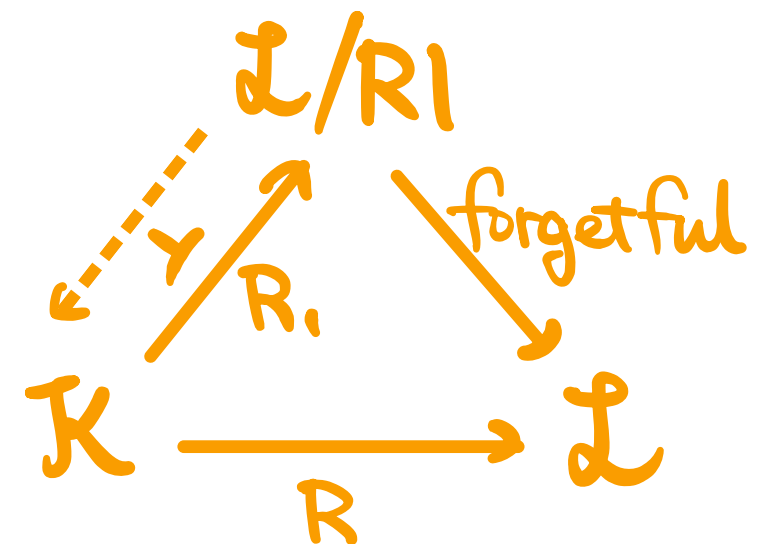
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# Enlarging the class of enrichment bases

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{ monoidal categories }  $\longrightarrow$  { multicategories }



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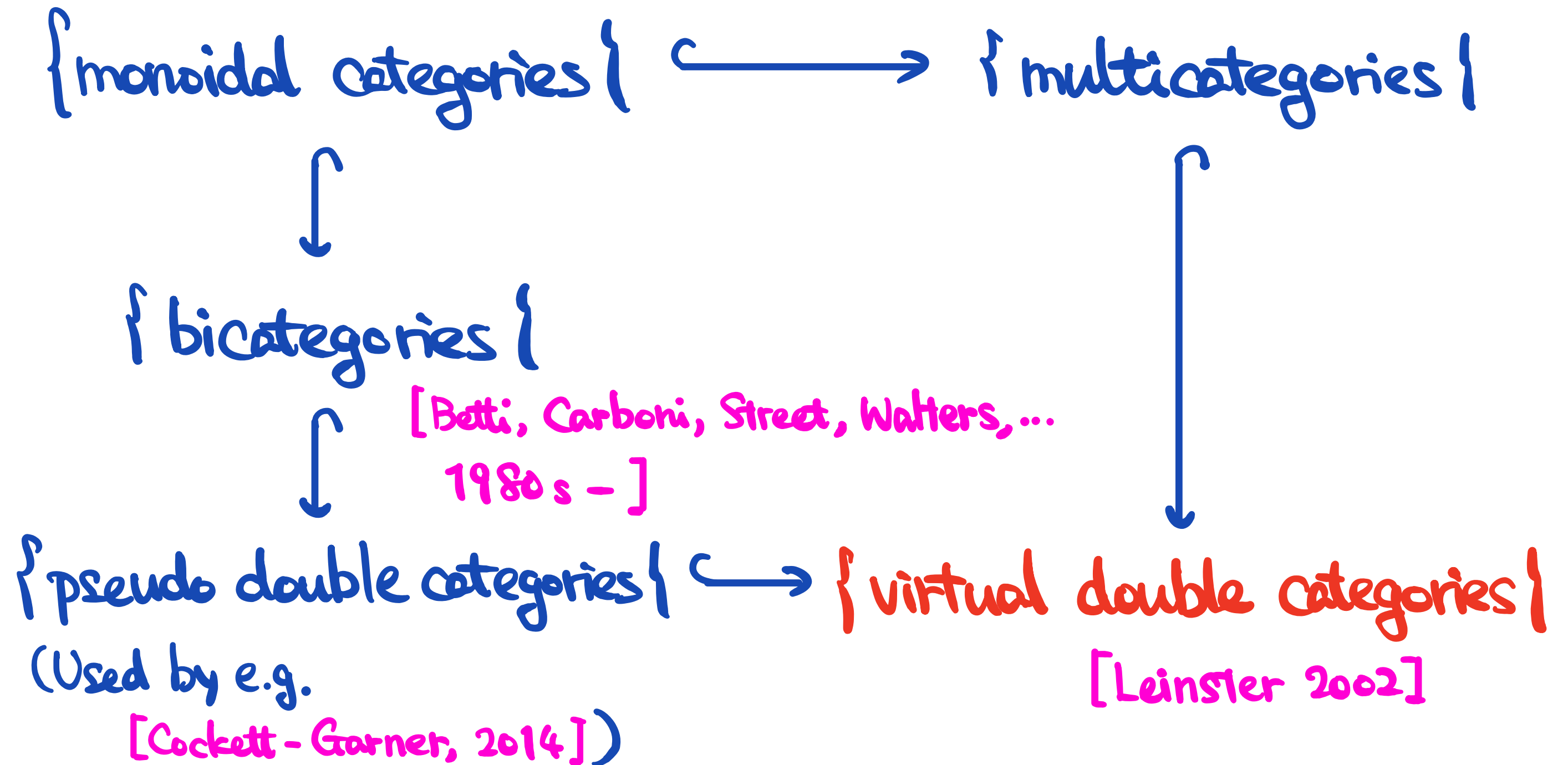
[Betti, Carboni, Street, Walters, ...  
1980s -]

{ pseudo double categories }

(Used by e.g.

[Cockett - Garner, 2014])

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$$\begin{array}{ccc} & \text{2-CAT/Enr(1)} & \\ \text{L} \swarrow & \nearrow \text{Enr}_! & \searrow \text{forgetful} \\ \text{VDBL} & \xrightarrow{\text{Enr}} & \text{2-CAT} \end{array}$$



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$$\begin{array}{ccccc}
 A_0 & \xrightarrow{\tau_1} & A_0 & \xrightarrow{\tau_2} & \dots & \xrightarrow{\tau_n} & A_n \\
 u \downarrow & & \alpha & & & & \downarrow u' \\
 B & \xrightarrow{\quad \quad \quad} & \Delta & \xrightarrow{\quad \quad \quad} & & & B'
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$$\begin{array}{ccccccc} & & A_0 & \xrightarrow{\tau_1} & A_1 & \xrightarrow{\tau_2} & A_2 & \xrightarrow{\tau_3} & A_3 \\ & \swarrow u & & \searrow \alpha_1 & & & \downarrow \alpha_2 & & \downarrow \alpha_3 \\ & B_0 & \xrightarrow{\quad} & B_1 & \xrightarrow{\quad} & B_2 & \xrightarrow{\quad} & B_3 \\ \downarrow v & & & & & & & & \downarrow v' \\ C & \xrightarrow{\quad \quad \quad \beta \quad \quad \quad} & C' \end{array}$$

$\mapsto$

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$A$ : virtual double category

An  $A$ -category  $C$  consists of:

- a set  $\text{ob } C$  of objects
- a function  $\text{ob } C \xrightarrow{|\cdot|^C} \text{ob } A$   
 $\downarrow \quad \xrightarrow{\quad} \quad \downarrow$   
 $x \quad \xrightarrow{\quad} \quad |x|^C$  ← The extent of  $x$ .
- $\forall x, y \in C$ , a horiz. mor.  
 $|x|^C \xrightarrow{C(x,y)} |y|^C \text{ in } A$
- multicells

$$\begin{array}{ccc} & |x|^C & \\ \eta_x^C \swarrow & & \searrow \eta_x^C \\ (x)^C & \xrightarrow{C(x,x)} & (x)^C \end{array}$$

$$\begin{array}{ccccc} |x|^C & \xrightarrow{C(x,y)} & |y|^C & \xrightarrow{C(y,z)} & |z|^C \\ \parallel & & \mu_{x,y,z}^C & & \parallel \\ |x|^C & \xrightarrow{C(x,z)} & |z|^C & & \end{array} \text{ in } A$$

satisfying the category axioms.

An  $A$ -functor  $C \xrightarrow{F} D$  consists of:

- a function  $\text{ob } C \xrightarrow{\text{ob } F} \text{ob } D$   
 $\downarrow \quad \xrightarrow{\quad} \quad \downarrow$   
 $x \quad \xrightarrow{\quad} \quad Fx$   
 $|x|^C \quad \xrightarrow{\quad} \quad |Fx|^D$
- $\forall x \in C$ , a vertical mor.  $|x|^C \downarrow F_x \text{ in } A$
- $\forall x, y \in C$ , a multicell

$$\begin{array}{ccccc} |x|^C & \xrightarrow{C(x,y)} & |y|^C & & \\ F_x \downarrow & & F_{xy} \downarrow & & F_y \downarrow \\ |Fx|^D & \xrightarrow{D(Fx,Fy)} & |Fy|^D & & \end{array} \text{ in } A$$

satisfying the functor axioms.

$\Rightarrow A\text{-Cat} : 2\text{-category}$

$$\begin{array}{ccc} \text{VDBL} & \xrightarrow{\text{Enr}} & 2\text{-CAT} : 2\text{-functor} \\ \downarrow & & \downarrow \\ A & \xrightarrow{\quad} & A\text{-Cat} \end{array}$$

$$\text{VDBL} \xrightleftharpoons[\text{Enr}_1]{\text{L}} 2\text{-CAT}(\text{Enr}(1))$$

What is  $\text{VDBL} \xrightleftharpoons[\text{Enr}_1]{\mathbb{L}} 2\text{-CAT}(\text{Enr}(1))$  ?

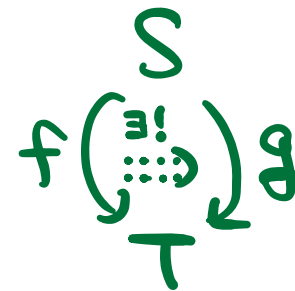


What is  $VDBL \xrightleftharpoons[Enr_1]{L} 2-CAT(Enr(1))$  ?

$Enr(1) = "1-Cat"$

What is  $\text{VDBL} \xrightleftharpoons[\text{Enr}_!]{\text{L}} 2\text{-CAT}(\text{Enr}(1))$  ?

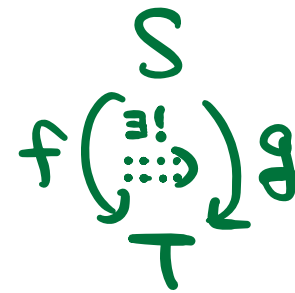
$\text{Enr}(1) = \text{"1-Cat"} = \text{Set}_{lc}$  : locally chaotic 2-category whose underlying category is  $\text{Set}$ .



What is  $\mathbf{VDBL} \xrightleftharpoons[\text{Enr}_!]{\mathbb{L}} \mathbf{2-CAT}(\text{Enr}(1))$  ?

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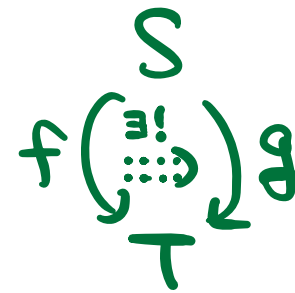
$$\text{Enr}_!(A) = \begin{pmatrix} \text{Enr}(A) \\ \downarrow \text{Enr}(!A) \\ \text{Enr}(1) \end{pmatrix}$$



What is  $\mathbf{VDBL} \xrightleftharpoons[\text{Enr}_1]{\perp} \mathbf{2-CAT}(\text{Enr}(1))$  ?

$\text{Enr}(1) = \text{"1-Cat"} = \text{Set}_{lc}$  : locally chaotic 2-category whose underlying category is  $\text{Set}$ .

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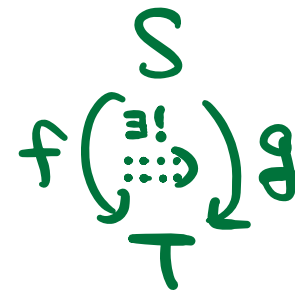


What is  $VDBL \xrightleftharpoons[\text{Enr}_!]{\mathbb{L}} 2\text{-CAT}/\text{Enr}(1)$  ?

$\text{Enr}(1) = "1\text{-Cat}" = \text{Set}_{lc}$  : locally chaotic 2-category whose underlying category is  $\text{Set}$ .

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Given  $\begin{pmatrix} \mathcal{K} \\ \downarrow G \\ \text{Set}_{lc} \end{pmatrix} \in 2\text{-CAT}/\text{Set}_{lc}$ ,  $LG \in VDBL$  is ...



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by  $2\text{-CAT} \xrightleftharpoons[\text{(-)}_{lc}]{\text{(-)}_o} \text{CAT}$

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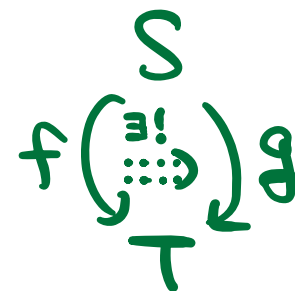
$$\text{Enr}_1(A) = \left( \begin{array}{c} \text{Enr}(A) \\ \downarrow \text{Enr}(!A) \\ \text{Enr}(1) \end{array} \right) = \left( \begin{array}{c} A\text{-Cat} \\ \downarrow \text{ob} \\ \text{Set}_{lc} \end{array} \right).$$

$$\left( \begin{array}{c} \mathcal{K}_0 \\ \downarrow \hat{G} \\ \text{Set} \end{array} \right) \text{ by } 2\text{-CAT} \xrightleftharpoons[\text{(-)}_{lc}]{\text{(-)}_0} \text{CAT}$$

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- obj.  $(K \in \mathcal{K}, \alpha \in GK)$

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- obj.  $(K \in \mathcal{K}, x \in GK)$

- v.mor.  $(k, x) \downarrow m LG = \begin{matrix} K \\ \downarrow \\ L \end{matrix} \text{ f m } \mathcal{K} \text{ s.t. } (Gf)x = y.$



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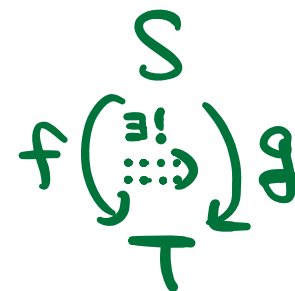
- obj.  $(K \in \mathcal{K}, x \in GK)$

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$$(LG)_{\text{vert}} = E(\hat{G})$$

$$f \begin{pmatrix} S \\ \exists! \downarrow \\ T \end{pmatrix} g$$

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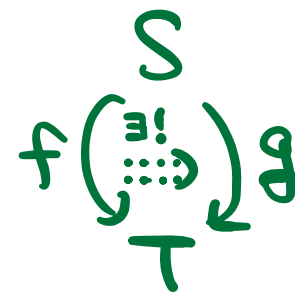
$$(LG)_{\text{vert}} = E(+)(\hat{G})$$

- horiz.mor

$$\{(k, x) \mapsto (k', x')\} =$$

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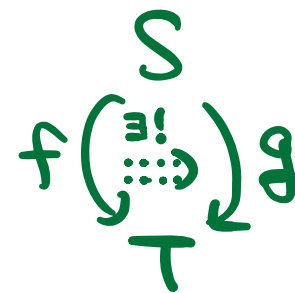
- v.mor.  $(k, x) \downarrow \text{in } LG = \begin{pmatrix} K \\ \downarrow f \text{ in } \mathcal{K} \text{ s.t. } (Gf)x = y. \\ L \end{pmatrix}$

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$$(LG)_{\text{vert}} = E(\perp)(\hat{G})$$

- horiz.mor

$$\{(k, x) \mapsto (k', x')\} = \begin{cases} \{*\} & \text{if } k = k' \\ \emptyset & \text{if } k \neq k'. \end{cases}$$

Abstract proof of  $\text{VDBL} \xrightarrow{\text{Enr}} \text{2-CAT} : \text{param. r. 2-adj.} :$

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Observation

| {param. right 2-adj.} is  
|

Abstract proof of  $\text{VDBL} \xrightarrow{\text{Enr}} \text{2-CAT} : \text{param. r. 2-adj.} :$   
Observation

|  $\{\text{param. right 2-adj.}\}$  is closed under composition  
|

Abstract proof of  $\text{VDBL} \xrightarrow{\text{Enr}} 2\text{-CAT} : \text{param. r. 2-adj.} :$   
Observation

[ {param. right 2-adj.} is closed under composition and  
contains {right 2-adj.} and {polynomial 2-functors}.



Abstract proof of  $\text{VDBL} \xrightarrow{\text{Enr}} 2\text{-CAT} : \text{param. r. 2-adj.} :$   
Observation

[  $\{\text{param. right 2-adj.}\}$  is closed under composition and  
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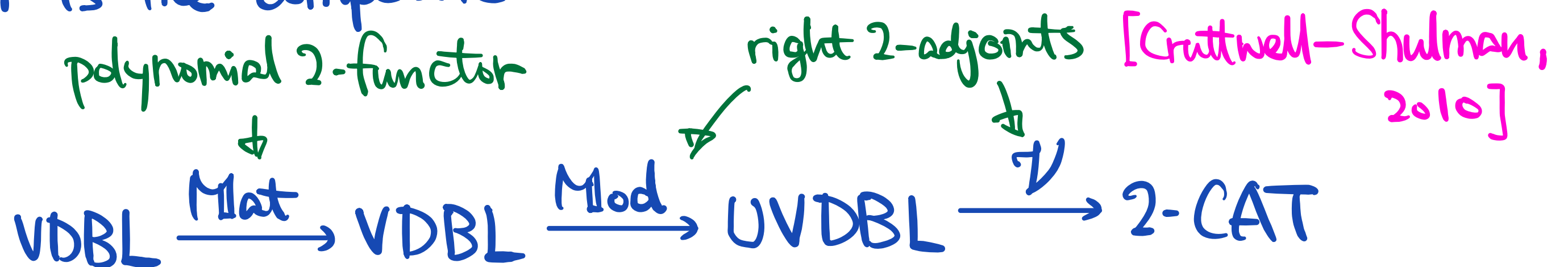
Enr is the composite

$$\text{VDBL} \xrightarrow{\text{Mat}} \text{VDBL} \xrightarrow{\text{Mod}} \text{UVDBL} \xrightarrow{\mathcal{V}} 2\text{-CAT}$$

Abstract proof of  $\text{VDBL} \xrightarrow{\text{Enr}} 2\text{-CAT} : \text{param. r. 2-adj.} :$   
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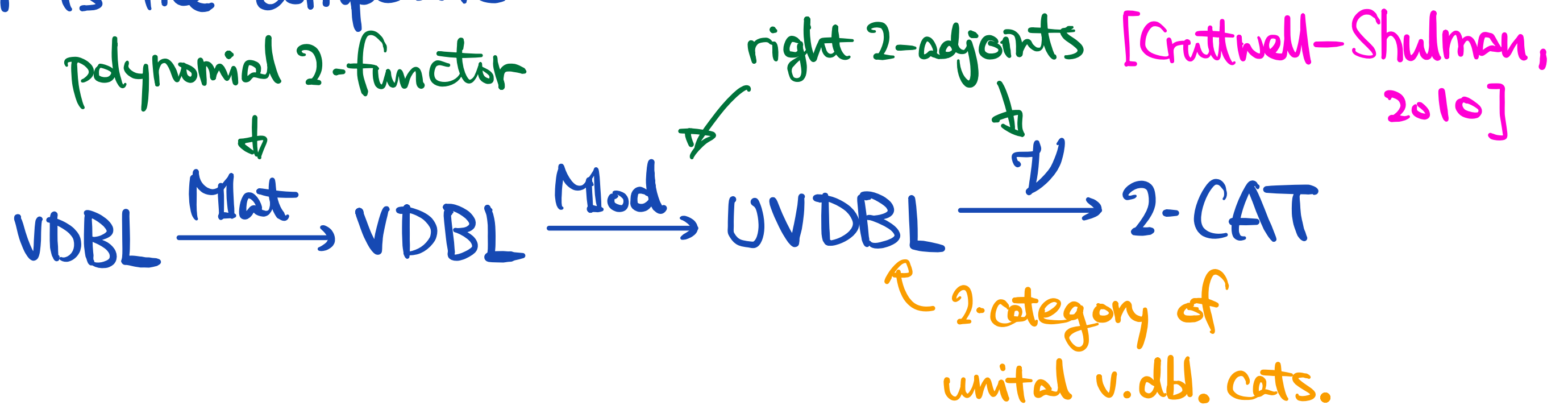
Enr is the composite  
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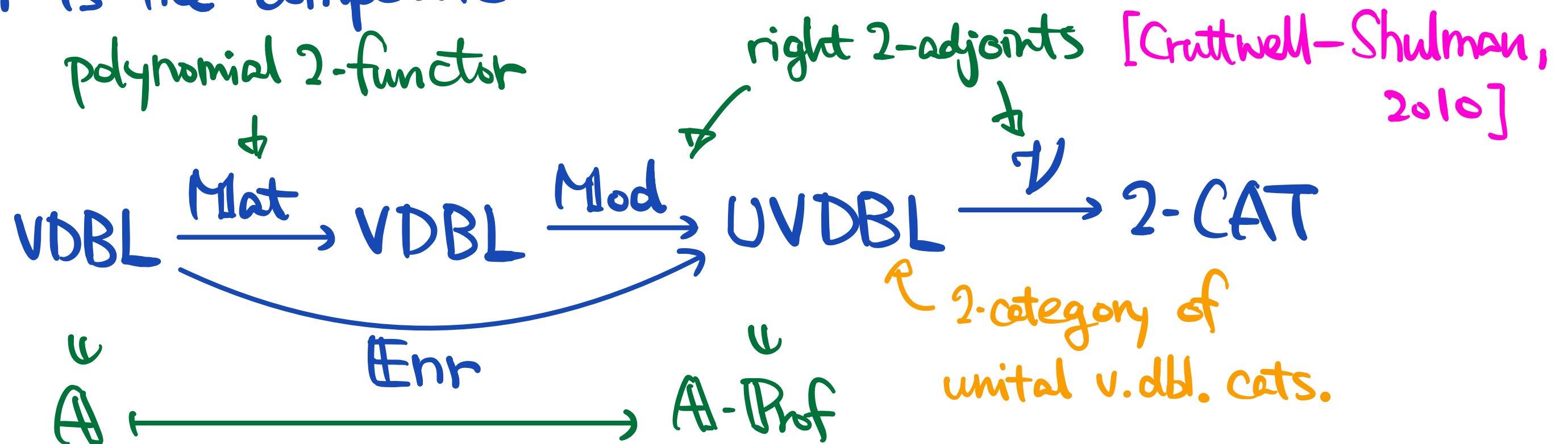
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Abstract proof of  $\text{VDBL} \xrightarrow{\text{Enr}} 2\text{-CAT} : \text{param. r. 2-adj.} :$   
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Enr is the composite  
 polynomial 2-functor



$$\begin{array}{ccc} \text{VDBL} & \xrightarrow{\text{Mat}} & \text{VDBL} \\ \downarrow & & \downarrow \\ \text{A} & \xrightarrow{\quad} & \text{A-Mat} \end{array}$$

$VDBL \xrightarrow{Mat} VDBL$

$\underbrace{A} \mapsto \underbrace{A}_{A-Mat}$

obj.  $(I: \text{set}, (A_i \in A)_{i \in I})$

$$\text{VDBL} \xrightarrow{\text{Mat}} \text{VDBL}$$

$$\underbrace{\text{A}} \xrightarrow{\quad} \underbrace{\text{A}}^{\text{e}}\text{-Mat}$$

$$\text{obj. } (I: \text{set}, (A_i \in A)_{i \in I})$$

$$\text{v.mon. } (I, (A_i)_i) \downarrow (J, (B_j)_j) \text{ is } \begin{array}{c} I \\ \downarrow f \\ J \end{array}, \left( \begin{array}{c} A_i \\ \downarrow u_i \\ B_{fi} \end{array} \in A \right)_{i \in I}$$

$$\text{VDBL} \xrightarrow{\text{Mat}} \text{VDBL}$$

$$\underbrace{\text{A}} \xrightarrow{\quad} \underbrace{\text{A-Mat}}$$

$$\left. \begin{array}{l} \text{obj. } (I: \text{set}, (A_i \in \mathbf{A})_{i \in I}) \\ \text{v.mor. } \begin{array}{c} (I, (A_i)_i) \\ \downarrow \\ (J, (B_j)_j) \end{array} \text{ is } \begin{array}{c} I \\ \downarrow f \\ J \end{array}, \left( \begin{array}{c} A_i \\ \downarrow u_i \\ B_{fi} \end{array} \right)_{i \in I} \end{array} \right\} (\text{A-Mat})_{\text{vert}} = \text{Fam}(\mathbf{A}_{\text{vert}}).$$



$$\text{VDBL} \xrightarrow{\text{Mat}} \text{VDBL}$$

$$\underbrace{\text{A}} \longmapsto \underbrace{\text{A-Mat}}$$

$$\text{obj. } (I: \text{set}, (A_i \in A)_{i \in I})$$

v.mor.

$$(I, (A_i)_i)$$

$$\downarrow$$

$$(J, (B_j)_j)$$

is

$$\begin{array}{c} I \\ \downarrow f \\ J \end{array}$$

$$\left( \begin{array}{c} A_i \\ \downarrow u_i \\ B_{fi} \end{array} \right)_{i \in I}$$

$$(A\text{-Mat})_{\text{vert}} = \text{Fam}(A_{\text{vert}}).$$

h.mor.

$$(I, (A_i)_i) \xrightarrow{(A_i \xrightarrow{T_{ij}} B_j \in A)_{i \in I, j \in J}} (J, (B_j)_j)$$

$$\text{VDBL} \xrightarrow{\text{Mat}} \text{VDBL}$$

$$\underbrace{\text{A}} \longmapsto \underbrace{\text{A-Mat}}$$

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$$\text{h.mor. } (I, (A_i)_i) \xrightarrow{(A_i \xrightarrow{T_{ij}} B_j \in A)_{i \in I, j \in J}} (J, (B_j)_j)$$

is the composite

$$\text{VDBL} \xrightarrow{(\text{Set}^*)_{\text{hc}} \times (-)} \text{VDBL}/(\text{Set}^*)_{\text{hc}} \xrightarrow{\Pi_P} \text{VDBL}/\text{Set}_{\text{hc}} \xrightarrow{\text{forgetful}} \text{VDBL}$$

$$\text{VDBL} \xrightarrow{\text{Mat}} \text{VDBL}$$

$$\underbrace{\text{A}} \xrightarrow{\quad} \underbrace{\text{A-Mat}}$$

$$\left. \begin{array}{l} \text{obj. } (I: \text{Set}, (A_i \in A)_{i \in I}) \\ \text{v.mor. } (I, (A_i)_i) \downarrow (J, (B_j)_j) \text{ is } \begin{array}{c} I \\ \downarrow f \\ J \end{array}, \left( \begin{array}{c} A_i \\ \downarrow u_i \\ B_{f(i)} \end{array} \right)_{i \in I} \end{array} \right\} (A\text{-Mat})_{\text{vert}} = \text{Fam}(A)_{\text{vert}}.$$

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induced by

$$1 \xleftarrow{!} (\text{Set}^*)_{\text{hc}} \xrightarrow{P} \text{Set}_{\text{hc}} \xrightarrow{!} 1$$

in VDBL.

$$\text{VDBL} \xrightarrow{\text{Mat}} \text{VDBL}$$

$$\underbrace{\text{A}} \xrightarrow{\quad} \underbrace{\text{A-Mat}}$$

$$\left. \begin{array}{l} \text{obj. } (I: \text{Set}, (A_i \in A)_{i \in I}) \\ \text{v.mor. } (I, (A_i)_i) \downarrow (J, (B_j)_j) \end{array} \right\} (A\text{-Mat})_{\text{vert}} = \text{Fam}(A)_{\text{vert}}$$

$$\text{is } \downarrow f, \left( \begin{array}{c} A_i \\ \downarrow u_i \\ B_{f(i)} \end{array} \right)_{i \in I} \in A$$

$$\text{h.mor. } (I, (A_i)_i) \xrightarrow{(A_i \xrightarrow{T_{ij}} B_j)_{i \in I, j \in J}} (J, (B_j)_j)$$

is the composite

$$\text{VDBL} \xrightarrow{(\text{Set}^*)_{\text{hc}} \times (-)} \text{VDBL}/(\text{Set}^*)_{\text{hc}} \xrightarrow{\Pi_P} \text{VDBL}/\text{Set}_{\text{hc}} \xrightarrow{\text{forgetful}} \text{VDBL}$$

induced by

$$1 \xleftarrow{!} (\text{Set}^*)_{\text{hc}} \xrightarrow{P} \text{Set}_{\text{hc}} \xrightarrow{!} 1$$

discrete opfibration  
 $\Rightarrow$  powerful (exponentiable) cf. [Bowler, 2011]

in VDBL.

The discrete opfibration  $(\mathbf{Set}_*)_{\mathbf{hc}} \xrightarrow{P} \mathbf{Set}_{\mathbf{hc}}$  is induced by  $\mathbf{Set}_* \xrightarrow{\text{forgetful}} \mathbf{Set}$  via  $\mathbf{CAT} \begin{array}{c} \xleftarrow{(-)_{\text{vert}}} \\ \perp \\ \xrightarrow{(-)_{\mathbf{hc}}} \end{array} \mathbf{VDBL}$ .

The discrete opfibration  $(\text{Set}_*)_{\text{hc}} \xrightarrow{P} \text{Set}_{\text{hc}}$  is induced by  $\text{Set}_* \xrightarrow{\text{forgetful}} \text{Set}$  via  $\text{CAT} \xrightleftharpoons[\text{(-)}_{\text{hc}}]{\text{(-)}_{\text{vert}}} \text{VDBL}$ .

Now:

- $\text{Set}_* \xrightarrow{\text{forgetful}} \text{Set}$  is a universal disc. opfib. (w/ small fibers) in CAT.

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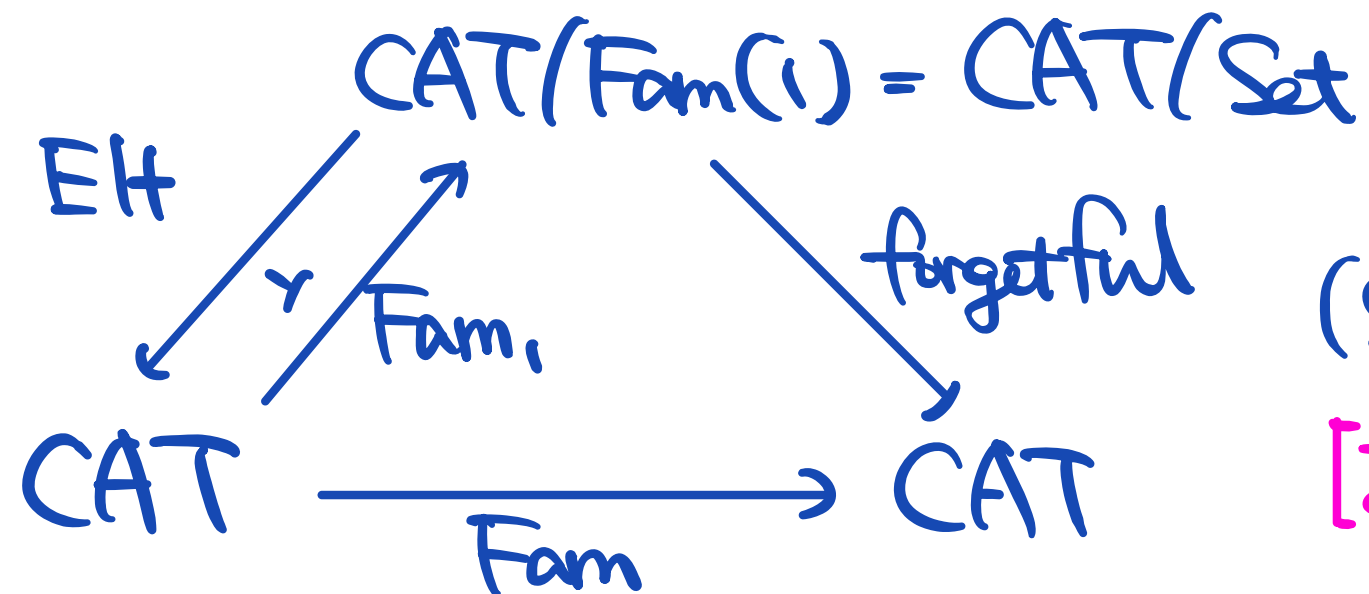
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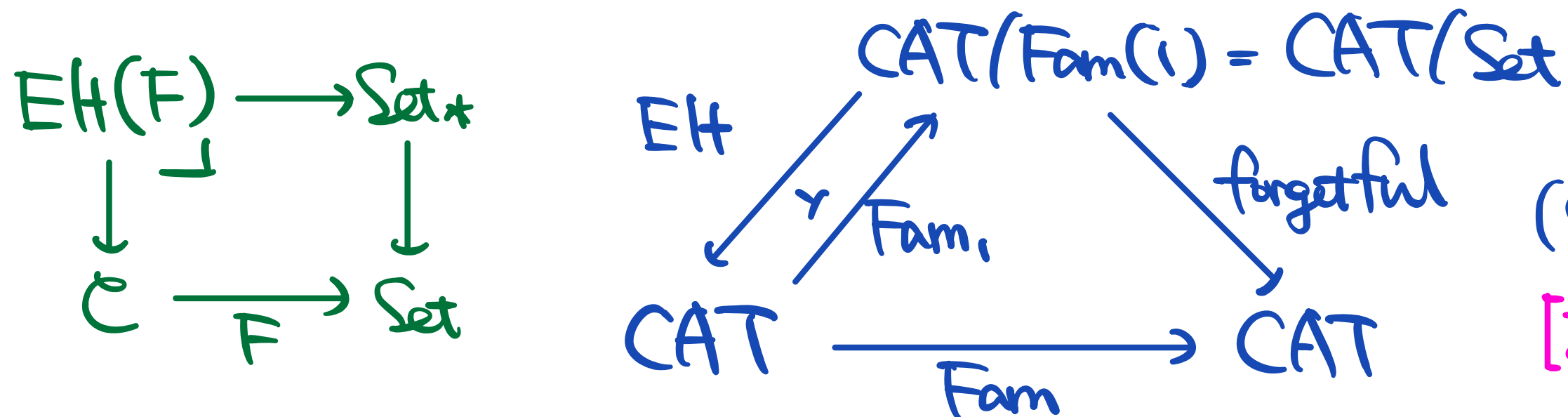


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