Categorical-algebraic characterisations of Lie algebras

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What makes Lie algebras so special?

Categorical characterisations of Lie algebras

Theorem (GM-Van der Linden, 2019)

Let $\mathcal{V}$ be a non-trivial variety of non-associative algebras over an infinite field.

If $\mathcal{V}$ satisfies the following property:

- *It is LACC*

then, $\mathcal{V}$ is the variety of Lie algebras.

Categorical characterisations of Lie algebras

Theorem (GM-Tsishyn-Vienne-Van der Linden, 2021)

Let $\mathcal{V}$ be a non-trivial variety of non-associative algebras over an infinite field.

If $\mathcal{V}$ satisfies the following property:

- *It's actions (or representations) are representable*

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Theorem (Deval-GM-Van der Linden, 2024)

Let $\mathcal{V}$ be a non-trivial variety of non-associative algebras over an infinite field.

If $\mathcal{V}$ satisfies the following property:

- *It admits a universal (split) Kaluzhnin-Krasner embedding theorem*

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Main Theorem

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Let $\mathcal{V}$ be a non-trivial variety of non-associative algebras over a field of characteristic zero.

If $\mathcal{V}$ satisfies the following two properties:

- *It is a 2-variety*
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- Associative algebras $x(yz) - (xy)z = 0$
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- Jordan algebras
- Trivial algebras $xy = 0$

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Over characteristic zero, every set of algebraic identities is equivalent to a multilinear one (all the monomials of the relations have exactly one x, one y, etc.)

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A variety of non-associative algebras is isomorphic to the class of algebras over a quotient of the free operad generated by a binary operation.

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- *$\mathcal{V}$ is a 2-variety*
- *$\mathcal{V}$ is algebraically coherent*
- *$\mathcal{V}$ is an Orzech-category of interest*
- *there exist $\lambda_1, \ldots, \lambda_{16}$ in $\mathbb{K}$ such that*

$$z(xy) = \lambda_1 y(zx) + \lambda_2 x(yz) + \lambda_3 y(xz) + \lambda_4 x(zy) + \lambda_5 (zx)y + \lambda_6 (yz)x + \lambda_7 (xz)y + \lambda_8 (zy)x$$

and

$$(xy)z = \lambda_9 y(zx) + \lambda_{10} x(yz) + \lambda_{11} y(xz) + \lambda_{12} x(zy) + \lambda_{13} (zx)y + \lambda_{14} (yz)x + \lambda_{15} (xz)y + \lambda_{16} (zy)x$$

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Theorem (Dotsenko-Umirbaev, 2023)

Suppose that the operad $\mathcal{O}$ encoding a variety of algebras $\mathcal{V}$ satisfies the following two properties:

- *for the reverse graded path-lexicographic ordering, each leading term of the reduced Gröbner basis of the corresponding shuffle operad $\mathcal{O}^f$ has the minimal leaf directly connected to the root,*
- *there exists an ordering for which each leading term of the reduced Gröbner basis of the corresponding shuffle operad $\mathcal{O}^f$ is a left comb whose second smallest leaf is a sibling of the minimal leaf.*

then, $\mathcal{V}$ is Nielsen-Schreier.

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Corollary

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Corollary

The variety of associative algebras is not Nielsen-Schreier

$$y(xz) - (yx)z$$

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Both properties are kind of a *Sandwhich* to each other.
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The 2-variety condition *likes* identities, while the Nielsen-Schreier *doesn't like* identities.

Both criteria meet in a thin line, and, with the help of some computational algebra, we end up realising that Lie algebras is the only possibility.

Final considerations

The condition (LACC) can be encoded as the preservation of coproducts of the $B\flat-$ functor, i.e. a category is LACC if the canonical morphism

$$B\flat X + B\flat Y \to B\flat(X + Y)$$

is an isomorphism.

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Both proofs have some similarities but they are not equivalent to each other.

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