Pretorsion Theories on $(\infty, 1)$-catégories

L. Grossman[1]
lucy.grossman@uclouvain.be

[1]Institut de recherche en mathématiques et physique
Université Catholique de Louvain, Louvain-la-Neuve, Belgique

International Category Theory Conference CT2025,
July 14, 2025,
Brno, Czechia

Our plan and similar work in the literature

In this talk, we will discuss a generalization of the notion of *pretorsion theory* to the context of infinity categories (here quasi-catégories). Many crucial ideas, including the following, have already been discussed in the literature.

- The concept of *pretorsion theory* as a generalization of Dickson's *torsion theories* (see [2]) has been developed extensively for 1-categories by Facchini, Finocchiaro, Gran, and others. See, for instance, [3] and [4].
- There is a notion of torsion theory, based on factorization systems, for stable $(\infty, 1)-$categories introduced in [5].
- In [7], there is a notion of torsion theory for bicategories under consideration.

Definition

(Definition 2.6 of [4]) Let $\mathbf{C}$ be a category. A *pretorsion theory* $(\mathbf{T}, \mathbf{F})$ on $\mathbf{C}$ consists of a pair of full, replete subcategories $\mathbf{T}$ and $\mathbf{F}$ such that for $\mathbf{Z} := \mathbf{T} \cap \mathbf{F}$, the following conditions are satisfied:

1. $hom_{\mathbf{C}}(T, F) = Triv_{\mathbf{Z}}(T, F)$ for every object $T \in \mathbf{T}$ and $F \in \mathbf{F}$.
2. For each object $B \in \mathbf{C}$ there exists a short $\mathbf{Z}-$exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with $A \in \mathbf{T}$ and $C \in \mathbf{F}$ (in other words, f is a $\mathscr{Z}$-kernel of g and g a $\mathscr{Z}$-cokernel of f).

Remark

- If $\mathbf{Z} = \varnothing$, this structure is that of a *torsion theory* (see [2]).

$(\infty, 1)$-Pretorsion Theories

We state the direct analogue for $(\infty, 1)$-categories, where all components are replaced by their $(\infty, 1)$ counterparts.

Definition:

Let $\mathscr{C}$ be an $(\infty, 1)$-category. A *pretorsion theory* on $\mathscr{C}$ is a triple of full, replete subcategories $(\mathscr{T}, \mathscr{F}, \mathscr{Z})$ such that:

1. $Hom_{\mathscr{C}}(T, F) = \mathscr{Z}\text{-}Triv(T, F), \forall T \in \mathscr{T}, F \in \mathscr{F}$, and
2. To every object $X \in \mathscr{C}$, one can associate a short $\mathscr{Z}-$exact sequence $TX \xrightarrow{\epsilon} X \xrightarrow{\eta} FX$.

Remarks:

- One usually takes $\mathscr{Z} := \mathscr{T} \cap \mathscr{F}$.
- If $\mathbf{Z} = \varnothing$, this structure is that of a *torsion theory* (see [2]).

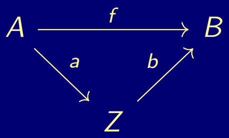
This all seems well and good, but the devil lies in the details.

$(\infty, 1)$-Pretorsion Theories - $\mathscr{Z}$-triviality

Fundamental challenge: defining $\mathscr{Z}$-triviality. In principle,

Definition:

A morphism $f : A \to B$ in $\mathscr{C}$ is $\mathscr{Z}$*-trivial* if, for $Z \in \mathscr{Z}$ the following diagram commutes:



,

i.e. $f \cong b \circ a$.

Our naïve attempt to lift this to $(\infty, 1)$-catégories was to define $\mathscr{Z}$-triv$(-.-)$ as a certain subfunctor of $Hom(-,-)$.

Remark:

In this definition, the $\mathscr{Z}$-kernels are epimorphisms and the $\mathscr{Z}$-cokernels are monomorphisms, severely restricting the collection of potential examples.

Pretorsion theories on QCs - sought-after example

Generalizing the example from [1], we consider the following.

Proposition:

Let $\mathscr{C}$ be an $(\infty, 1)$-category. Denote by $Seg(\mathscr{C})$ the Segal space objects of $\mathscr{C}$, by $GpdSeg(\mathscr{C})$ the Segal space groupoid objects of $\mathscr{C}$, by $Seg\hat{G}pd(\mathscr{C})$ the complete Segal space groupoid objects of $\mathscr{C}$, and $S\hat{e}g(\mathscr{C})$ the complete Segal space objects of $\mathscr{C}$.
Then, $(\mathscr{T}, \mathscr{F}, \mathscr{Z}) = (SegGpd(\mathscr{C}), S\hat{e}g(\mathscr{C}), Seg\hat{G}pd(\mathscr{C})$ forms a pretorsion theory on $Seg(\mathscr{C})$.

Remark:

This does not hold using our previous definition, as namely the short exact sequence associated to an object $X \in Seg(\mathscr{C})$ is of the form

$$X^c \xrightarrow{core} X \xrightarrow{cmpl.} \bar{X},$$

with *core* the Segal core functor and *cmpl.* that of Segal completion.

Problem and Potential Fix:

The map $X \xrightarrow{cmpl.} \bar{X}$ is not an epimorphism. We would like this to be an example of an $(\infty, 1)$-pretorsion theory, so we shall adapt the definition to accomodate this.

One Potential Fix:

Define $\mathscr{Z}-$triviality instead in terms of the coend

$$\int^{z \in \mathscr{Z}} Hom_{\mathscr{C}}(X, Z) \times Hom_{\mathscr{C}}(Z, Y).$$

This at least encapsulates the factorization that $\mathscr{Z}-$triviality implies.

Remarks:

- This makes the proposed PTT on $Seg(\mathscr{C})$ an actual example.
- Another option is to define $\mathscr{Z}\text{-}Triv(-,-)$ as some other functor.

Remarks:

- Our original definition of PTT on an $(\infty, 1)$-category was developed to directly lift the $1-$categorical notion to that setting. It does this, and once one passes to $h\mathscr{C}$, one obtains the classical $1-$categorical notion and its properties.
- In the case that one takes $\mathscr{Z} = \varnothing$ and works in a stable category, one arrives at a notion of torsion theory. This will be coherent with the notion developed in [5].
- By truncating our $(\infty, 1)-$pretorsion theories using the original definition, we obtain the bicategorical ones of [7] up to a uniqueness condition.

References

Francis Borceux, Federico Campanini, Marino Gran, and Walter Tholen.
Groupoids and skeletal categories form a pretorsion theory in Cat, 2023.

Spencer E. Dickson.
A torsion theory for abelian categories.
Transactions of the American Mathematical Society, 121:223–235, 1966.

Alberto Facchini and Carmelo Finocchiaro.
Pretorsion theories, stable category and preordered sets, 2019.

Alberto Facchini, Carmelo Finocchiaro, and Marino Gran.
Pretorsion theories in general categories.
Journal of Pure and Applied Algebra, 225(2):106503, 2021.

D. Fiorenza and F. Loregian.
t-structures are normal torsion theories.
Applied Categorical Structures, 24(2):181–208, Apr 2016.

Lucy Grossman.
Théories de prétorsion sur les quasi-catégories. forthcoming.
2024.

Mariano Messora and Enrico Vitale.
Bi-torsion theories - slides on an ongoing project, 2024.