Actions of partial groups and the higher Segal conditions

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Example

$[-3,3] \cap \mathbb{Z}$:

$$+(1,-2) = -1 \qquad +(2,-1,2) = 3$$
$$+(2,2) = ? \qquad +(-1,2,2) = ?$$

Example (Adem, Cohen, Torres Giese)

G group $\rightsquigarrow G_{\text{com}}$ partial group with same elements

$$\bullet(g_1, \dots, g_n) = \begin{cases} g_1 \cdots g_n & \text{if } g_i g_k = g_k g_i \text{ for all } i, k \\ ? & \text{otherwise} \end{cases}$$

The nerve theorem for categories

The nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful.

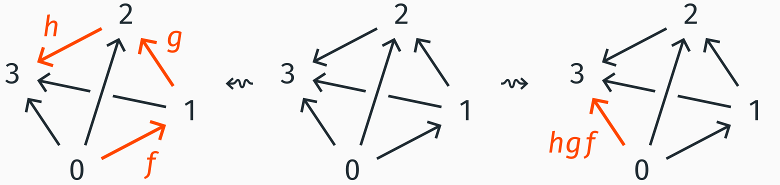
Moreover, the following are equivalent for $X \in \mathsf{sSet}$:

1. $X \cong NC$ for some $C \in \mathsf{Cat}$
2. the *Segal map* $\mathrm{sp}_n : X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is a bijection for all $n \geq 2$
3. $$\begin{array}{ccc} X_n & \xrightarrow{d_n} & X_{n-1} \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_{n-1} & \xrightarrow[d_{n-1}]{} & X_{n-2} \end{array}$$ is a pullback for all $n \geq 2$

The Segal map

$$\mathrm{sp}_n : X_n \to X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \times_{X_{\{2\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}$$

$$= X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 =: \mathbf{W}X_n \subseteq \prod_{i=1}^{n} X_1$$



n-ary composition: $\mathbf{W}X_n \xleftarrow{\cong} X_n \to X_1$

The nerve theorem for groupoids

$\mathbf{\Sigma}$ has objects $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$, all functions as maps.
With $[n]$ considered as a chaotic groupoid, $\mathbf{\Sigma} \subseteq \mathrm{Gpd}$.
$\mathrm{Sym} = \mathrm{Fun}(\mathbf{\Sigma}^{\mathrm{op}}, \mathrm{Set})$ = category of *symmetric (simplicial) sets*.

The nerve functor $N\colon \mathrm{Gpd} \to \mathrm{Sym}$ is fully faithful.

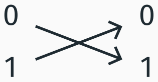
Moreover, the following are equivalent for $X \in \mathrm{Sym}$:

1. $X \cong NC$ for some $C \in \mathrm{Gpd}$
2. the *Segal map* $\mathrm{sp}_n : X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is a bijection for all $n \geq 2$

Why are symmetric sets a great home for groupoids?

Finding inverse:

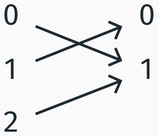
$$[1] \xrightarrow{\text{flip}} [1] \qquad\qquad X_1 \longrightarrow X_1$$



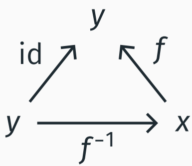
$$\rightsquigarrow \quad f \longmapsto f^{-1}$$

Checking it works:

$$[2] \xrightarrow{\text{fold}} [1] \qquad\qquad X_1 \longrightarrow X_2$$



$$\rightsquigarrow \quad f \mapsto$$



Definition
A symmetric set X is *spiny* if the Segal maps

$$\mathrm{sp}_n : X_n \to \mathbf{W}X_n \subseteq X_1^{\times n}$$

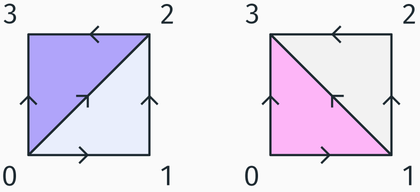
are injections for all $n \geq 2$. Also say X is a *partial groupoid*.

Example
A symmetric subset of the nerve of a groupoid is spiny.

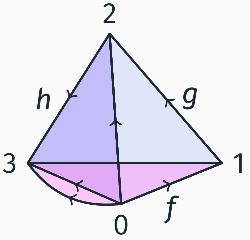
Definition/Theorem (Chermak, González, H–Lynd)
A *partial group* is a reduced partial groupoid (i.e. $X_0 = *$).

Not every partial group embeds in a group

Platonically non-associative partial groupoid



glue together to obtain a 2-dimensional partial groupoid:



Action of partial group: motivation

A functor $E \xrightarrow{p} B$ between categories is a *discrete opfibration* if:

$$\begin{array}{ccc} e & & \\ \vdots & & \\ p(e) & \longrightarrow & b \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} e & \xrightarrow{\exists!} & e' \\ \vdots & & \vdots \\ p(e) & \longrightarrow & b \end{array}$$

Category of elements / Grothendieck construction:

$$\int : \mathrm{Fun}(B, \mathrm{Set}) \simeq \mathrm{dOpFib(B)} \subseteq \mathrm{Cat}_{/B}$$

objects $(b \in B, x \in F(b))$
morphisms $f_x : (b, x) \to (b', F(f)x)$ for $f : b \to b'$

Action of partial group (or partial groupoid)

Definition

For $X \in \mathrm{Sym}$ and $x \in X_0$, let $\bigstar(x)$ be the collection of all simplices that start at x:

$$\begin{array}{ccc} \bigstar(x)_n & \longrightarrow & X_n \\ \big\downarrow & \lrcorner & \big\downarrow \text{start} \\ \{x\} & \longrightarrow & X_0 \end{array}$$

Definition

A map $p\colon E \to B$ in Sym is *star injective* if

$$\bigstar(e) \to \bigstar(p(e))$$

is injective for each $e \in E_0$

Star injective maps as actions

Lemma
If B is spiny and $p\colon E \to B$ is star injective, then E is spiny.

Think of p as a (partial) action of B on the set E_0:

$$f = \left(b_0 \xrightarrow{f_1} b_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} b_n \right) \in B_n \qquad e = e_0 \in p^{-1}(b_0)$$

$$\begin{array}{ccccccc} e_0 & \xrightarrow{\tilde{f}_1} & e_1 & \xrightarrow{\tilde{f}_2} & \cdots & \xrightarrow{\tilde{f}_n} & e_n \\ \vdots & & \vdots & & & & \vdots \\ b_0 & \xrightarrow{f_1} & b_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & b_n \end{array} \qquad f \cdot e := \begin{cases} e_n \\ ? \end{cases}$$

Example (Exel, Kellendonk–Lawson)

G group, S a G-set, and $U \subseteq S$ a subset.
This is a *partial action* of G on U.

$$E \xhookrightarrow{\text{f.f.}} E' \xrightarrow{\text{dopfib}} G$$

$$\text{ob}E = U \qquad\qquad \text{ob}E' = S$$

$E \to G$ is a star injective map of groupoids.

Example

Passing to symmetric sets via the nerve, let $L_U(G) \subseteq G$ be the image of the above map in Sym.
If $U \neq \varnothing$ then $L_U(G)$ is a partial group, and $E \to L_U(G)$ is star inj.

Example: Punctured p-local groups

G group with nontrivial Sylow p-subgroup S for some prime p.
U = nontrivial elements of S with partial conjugation action.

$L_U(G)$ is a p-local punctured group (Henke–Libman–Lynd).

The *localities* of Chermak are partial groups generalizing this.

Characteristic actions

Definition

A star injective map $p\colon E \to B$ is *characteristic* if

- p is surjective
- E is a groupoid

Example

$E \to L_U(G)$ is characteristic

Theorem

Every partial groupoid admits a characteristic action.

Proof.

$$\coprod_{n \geq 0} \coprod_{\mathrm{nd} L_n} \hom_{\mathbf{\Sigma}}(-, [n]) \to L \text{ is characteristic.}$$

□

Higher Segal conditions and the discrete geometry of actions

Higher Segal spaces (Dyckerhoff & Kapranov; Walde)

$$\begin{array}{ccc} X_n & \xrightarrow{d_0} & X_{n-1} \\ {\scriptstyle d_n}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_{n-1}} \\ X_{n-1} & \xrightarrow[d_0]{} & X_{n-2} \end{array}$$

lower 1-Segal

$$\begin{array}{ccc} X_n & \xrightarrow{d_i} & X_{n-1} \\ {\scriptstyle d_0}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_0} \\ X_{n-1} & \xrightarrow[d_{i-1}]{} & X_{n-2} \end{array}$$

$2 \leq i \leq n-1$

upper 2-Segal

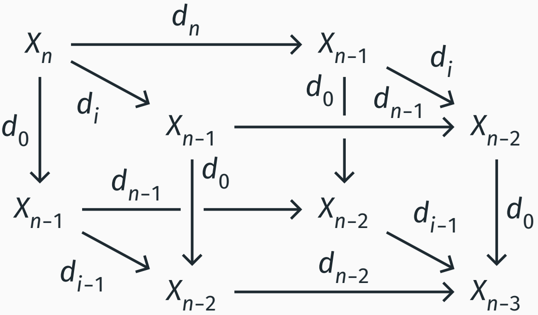
$$\begin{array}{ccc} X_n & \xrightarrow{d_j} & X_{n-1} \\ {\scriptstyle d_n}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_{n-1}} \\ X_{n-1} & \xrightarrow[d_j]{} & X_{n-2} \end{array}$$

$1 \leq j \leq n-2$

lower 2-Segal

Higher Segal spaces (Dyckerhoff & Kapranov; Walde)

lower 3-Segal:



$2 \leq i \leq n-2$

Theorem
For symmetric sets, the lower $(2k-1)$*-Segal, lower* $2k$*-Segal, upper* $2k$*-Segal, and upper* $(2k+1)$*-Segal conditions coincide.*

The degree

Theorem
For symmetric sets, the lower $(2k-1)$*-Segal, lower* $2k$*-Segal, upper* $2k$*-Segal, and upper* $(2k+1)$*-Segal conditions coincide.*

Definition
The *degree* of a symmetric set X is the least $k \geq 1$ such that X is lower $(2k-1)$-Segal.

Example
Degree one partial groupoids are just groupoids.

Closure space of action

Definition
$p\colon E \to B$ characteristic
E_0 is a closure space with generating closed sets

$$D(f) = \{x \in E_0 \mid f \in B_n \text{ acts on } x\}.$$

Arbitrary closed sets are intersections of these

Definition
The *Helly number* of $p\colon E \to B$ is the Helly number of E_0
i.e. $h(p) = \sup(n)$ ranging over those $n \in \mathbb{N}$ for which there is a family $(A_1, \dots, A_n)$ of closed sets with

$$\bigcap_{i=1}^{n} A_i = \varnothing \qquad \bigcap_{i \neq k} A_i \neq \varnothing \quad k = 1, \dots, n$$

Main Theorem

Definition

The *degree* of a symmetric set X is the least $k \geq 1$ such that X is lower $(2k-1)$-Segal.

Theorem

$p\colon E \to B$ characteristic

- *If B is not a groupoid, then* $\deg(B) \leq h(p)$
- *If additionally E_0 is artinian, then* $h(p) = \deg(B)$

Calculations: punctured Weyl groups

Φ a root system
$W = W(\Phi)$ the Weyl group
$\Gamma \subset \Phi$ a set of positive roots
$L = L_\Gamma(W)$ (punctured Weyl group: $L_1 = W \setminus \{w_0\}$)

Φ	$\deg(L)$	Φ	$\deg(L)$
A_n	$\lfloor (n+1)^2/4 \rfloor$	B_n/C_n	$\binom{n}{2}+1$
D_n	$\binom{n}{2}$	F_4	6
E_6	16	G_2	2
E_7	27	$I_2(m)$	2
E_8	36		