Natural Equivalences & Weak Invertibility of Higher-Categorical Contexts

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TalTech

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Let $e : u \Rightarrow v$ be a $(k+1)$-cell in a higher category.

If e is an equivalence, we expect the following:

$$- *_k e : \mathrm{Hom}(u', u) \longrightarrow \mathrm{Hom}(u', v)$$

$$e *_k - : \mathrm{Hom}(v, v') \longrightarrow \mathrm{Hom}(u, v')$$

are both equivalences of higher categories

In elementary terms:

Every well-formed equation of the form

$$x *_k e \overset{?}{=} b$$

$\dim b = n > k$

has a weak solution

$$a *_k e \simeq b$$

exhibited by $(n+1)$-equivalence

which is weakly unique

$$a' *_k e \simeq b \implies a' \simeq a$$

(and the dual)

A cell $e : u \Rightarrow v$ is weakly invertible if

$$\exists\ e^* : v \Rightarrow u,$$

$$h : e * e^* \Rightarrow 1_u, \qquad h' : 1_v \Rightarrow e^* * e$$

s.t. h, h' are weakly invertible

(This is a coinductive definition; formally, it determines a set $\mathrm{inv}\ X$ as a g.f.p. of an operator on $\Phi(\mathrm{cell}\ X)$)

Is this a good notion of equivalence?

The proof for strict ω-categories is essential in the characterisation of equivalences in the folk model structure

This proof

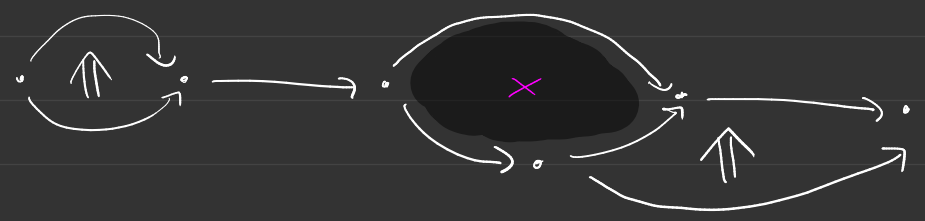
1. is surprisingly non-trivial,
2. uses strictness of units in an essential way

   ("the case $e = 1_u$ is easy")

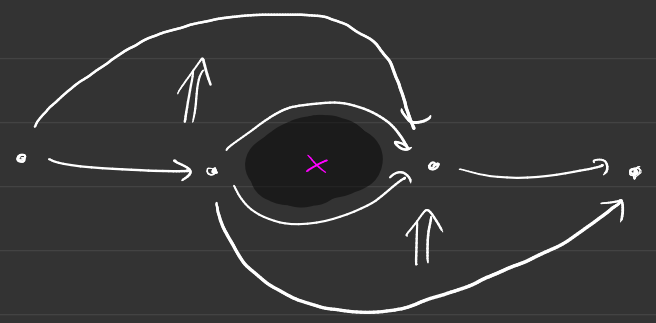
We study this problem in the theory of diagrammatic sets:

- a "slight weakening" of ω-categorical pasting
- boundaries of $(k+1)$-cells are restricted to certain "topologically nice" diagrams (round diagrams)
- expressiveness is restored by an algebra of weak units with which we can "pad" diagrams
- can internalise weak invertibility wrt round diagrams

A context is a "pasting diagram with a hole"



A round context is a round diagram with a hole



— We generalise the problem to equations

$$E\,x \doteq b$$

where E is a round context suitably "built up from weakly invertible round diagrams"

— We define an elementary, coinductive notion of natural equivalence of round contexts

$$\vartheta : C \simeq D \quad \text{with components} \quad \vartheta a : Ca \simeq Da$$

— We prove that every "weakly invertible round context" E has a "natural weak inverse" E^* s.t. $EE^* \simeq -$, $E^*E \simeq -$

— This implies that

$$E \times \overset{?}{=} b$$

has the weakly unique weak solution $E^* b$

References: (all by Chanavat & Hadzihasanovic)

2024 Equivalences in diagrammatic sets

main development

Model structures for diagrammatic (∞, n)-categories

used to characterise weak eqs. in a model of weak higher categories

2025 Semi-strictification of (∞, n)-categories

a variant of this theory plays a role