Generalised ultracategories and conceptual completeness of geometric logic

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Ultracategories

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- These are reconstruction theorems, allowing to recover a coherent theory via some syntactic object from its category of models (semantics).
- Among several notion of functors between ultracategories we distinguish between ultrafunctors (ultraproduct preserving functors), and left ultrafunctors (functors with comparison maps $\sigma_\mu : F(\int_I M_i d\mu) \to \int_I F(M_i) d\mu$)

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- In Lurie's case, the proof "relies" on the fact that Compact Hausdorff spaces are dense inside the 2-category of ultracategories, hence the ultrastructure can be recovered by left ultrafunctors from compact Hausdorff spaces.
- We wish to extend Lurie's result to any topos with enough points, but this requires a new language.

Generalised ultracategories

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- We use the same idea in a generalised ultracategory, but we replace the collection of objects $(M_i)_{i=1}^n$ by a formal ultraproduct of a collection of objects $(M_i)_{i \in I}$, with respect to an ultrafilter μ:

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- Similar to ultracategories, we may define left ultrafunctors between generalised ultracategories, and their natural transformations, giving rise to a 2-category of generalised ultracategories.

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- These are the models of the geometric theory classified by the topos.
- Given a site of definition $(C, \mathcal{J})$ of a topos E, the points of this topos are equivalent to $\mathcal{J}$-continuous flat functors from $(C, \mathcal{J})$ to Set, where $(C, \mathcal{J})$ is a site of definition of this topos.

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- The ultraproduct of points is not necessarily a point, but still makes sense inside $\mathrm{Fun}(C, \mathsf{Set})$.
- So we may define the generalised ultrastructure by $\mathrm{Hom}(N, \int_I M_i d\mu)$ inside $\mathrm{Fun}(C, \mathsf{Set})$.

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- The main theorem is equivalent to saying that the full subcategory of the category of generalised ultracategories whose objects are points of toposes, is equivalent to the 2-Category of Toposes (with enough points).
- We can deduce Lurie's conceptual completeness from the last theorem, by restricting our attention to the case where E is a coherent topos.

Proof Description

- We start by doing the proof in the case E is a topological space and $E^{'}$ is $S[\mathcal{O}]$ the classifying topos of theory of objects, in other words we show that for any space X, we have $\mathrm{Lult}(X, Set) \simeq Sh(X)$.

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- Finally we can show the theorem in its full generality, by paralleling two facts:
- Topological spaces are dense inside generalised ultracategories, i.e. the generalised ultrastructure is fully determined by left ultrafunctors from topological spaces.
- The next ingredient we need is a theorem by Moerdijk and Butz, stating that any topos with enough points is a colimit of a topological groupoid.