Unfolding of symmetric monoidal (∞, n)-categories

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CT2025, Brno
16 July 2025

Introduction

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- duals and adjoints in (symmetric monoidal) (∞, n)-categories
- cocartesian fibrations and straightening

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Informally, an (∞, n)-category has i-morphisms for all i, but they are all invertible for $i > n$, and composition is only associative up to coherent choice of higher invertible morphisms.

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A **symmetric monoidal** (= **s.m.**) (∞, n)-category is a commutative monoid in $\mathrm{Cat}_{(\infty,n)}$.

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and **adjoints** in the homotopy 2-category, i.e. f is left adjoint to g if there are 2-morphisms $\mathrm{id} \to gf$, $fg \to \mathrm{id}$ such that

$$f \to fgf \to f \;\simeq\; \mathrm{id}_f, \quad g \to gfg \to g \;\simeq\; \mathrm{id}_g\,.$$

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A morphism $f: x \to y$ in $\mathcal{E}$ is **p-cocartesian** if for all $z \in \mathcal{E}$, the commutative square

$$\begin{array}{ccc} \mathcal{E}(y,z) & \xrightarrow{f^*} & \mathcal{E}(x,z) \\ \downarrow & & \downarrow \\ \mathcal{B}(py,pz) & \xrightarrow[p(f)^*]{} & \mathcal{B}(px,pz) \end{array}$$

is a pullback.

Cocartesian fibrations

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Warning: Only the correct definition when $\mathcal{B}$ is an $(\infty, 1)$-category (not an (∞, n)-category)!

Straightening

Theorem (Lurie ($n = 1$), Nuiten, Blans–Blom)

There is a natural equivalence of $(\infty, 1)$-categories

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where $\mathrm{Cocart}_n(\mathcal{B})$ is the sub-$(\infty, 1)$-category of $\mathrm{Cat}_{(\infty,n)/\mathcal{B}}$ with

- objects the cocartesian fibrations over $\mathcal{B}$,
- and morphisms the functors that preserve cocartesian morphisms over $\mathcal{B}$.

Basic idea

Idea (Lurie)

A s.m. (∞, n)-category $\mathcal{C}$ with duals and certain adjoints can be reconstructed from the pullback squares of s.m. $(\infty, 1)$-categories

$$\begin{array}{ccc} \mathfrak{c}_1\Omega^k\mathcal{C} & \longrightarrow & \mathfrak{c}_1\mathbb{P}\Omega^{k-1}\mathcal{C} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{c}_1\Omega^{k-1}\mathcal{C} \end{array} \qquad (k = 1, \ldots, n-1).$$

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- $\mathbb{P}\mathcal{C} \to \mathfrak{c}_1\mathcal{C}$ is the cocartesian fibration for the functor

$$\mathcal{C}(\mathbb{1}, -)\colon \mathfrak{c}_1\mathcal{C} \to \mathrm{Cat}_{(\infty, n-1)},$$

with objects $\mathbb{1} \to X$, morphisms $\begin{array}{c} \mathbb{1} \\ \swarrow \Rightarrow \searrow \\ X \longrightarrow Y, \end{array}$ ...

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- and $\Omega\mathcal{C} = \mathcal{C}(\mathbb{1}, \mathbb{1})$, $\Omega^2\mathcal{C} = \Omega\mathcal{C}(\mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) = \mathcal{C}(\mathbb{1}, \mathbb{1})(\mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$, etc.

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A **categorical chain complex** of length ℓ is a sequence of pullbacks of s.m. $(\infty, 1)$-categories with duals

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Analogy with chain complexes of abelian groups $(C_\bullet, \partial)$: take $Z_k = \ker \partial \colon C_k \to C_{k-1}$ and describe the chain complex by fibre sequences

$$Z_k \to C_k \xrightarrow{\partial} Z_{k-1}.$$

Why is this interesting?

- Can be used to inductively construct s.m. (∞, n)-categories and s.m. functors among them from much simpler data, e.g. extended TQFTs
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- We can construct spectra from s.m. (∞, n)-categories by inverting everything; chain complex description gives some information about this (and this extends to unbounded chain complexes).

Precise statement

Theorem (Lurie, H.–Nikolaus)

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- A s.m. (∞, n)-category $\mathcal{C}$ is **left k-rigid** if $\mathcal{C}$ has duals for objects, all morphisms $\mathbb{1} \to X$ have left adjoints (if $k > 1$), and $\Omega\mathcal{C}$ is left $(k - 1)$-rigid. **Left rigid** $=$ left n-rigid.

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- A categorical chain complex is **complete** if for each k the canonical map

$$\begin{array}{ccc} \mathcal{Z}_{k,\mathbb{1}/} & \longrightarrow & \mathcal{C}_{k+1} \\ & \searrow \quad \swarrow_{\partial} & \\ & \mathcal{Z}_k & \end{array}$$

gives equivalences of ∞-groupoids on fibres

$$\mathcal{Z}_k(\mathbb{1}, X) \simeq \mathcal{C}^{\simeq}_{k+1,X}.$$

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- $\mathcal{A}_i$ is a left rigid s.m. (∞, i)-category
- and $\mathcal{A}_i \to \mathcal{A}_{i+1}$ is $(i - 1)$-surjective (surjective on objects, morphisms, etc., up to $(i - 1)$-morphisms.)

The case $n = 2$

- Want to recover a s.m. $(\infty, 2)$-category $\mathcal{C}$ with duals from

$$\mathbb{P}\mathcal{C} \to \mathfrak{c}_1\mathcal{C},$$

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so this functor knows all mapping $(\infty, 1)$-categories of $\mathcal{C}$.

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- **Idea:** Recover $\mathcal{C}$ by changing this enrichment along the lax s.m. functor $\mathcal{C}(\mathbb{1}, -)$.
- We prove that in general we can recover a closed s.m. $\mathcal{V}$-$(\infty, 1)$-category $\mathcal{D}$ from the lax s.m. functor $\mathcal{D}(\mathbb{1}, -) \colon U\mathcal{D} \to \mathcal{V}$ by changing enrichment from $\overline{U\mathcal{D}}$.

Two-step unfolding

- **Upshot:** Can recover a s.m. (∞, n)-category $\mathcal{C}$ with duals from $\mathbb{P}\mathcal{C} \to \mathfrak{c}_1\mathcal{C}$ (where $\mathbb{P}\mathcal{C}$ is an $(\infty, n-1)$-category).

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- But $\mathbb{P}^2\mathcal{C}$ won't have duals, since $\mathbb{P}\mathcal{C}$ can't have many adjoints (must live over adjoints = equivalences in $\mathfrak{c}_1\mathcal{C}$) — so can't iterate this argument.

Two-step unfolding

Idea: The triangle

$$\begin{array}{ccc} \mathbb{P}^2\mathcal{C} & \longrightarrow & \mathfrak{c}_1\mathbb{P}\mathcal{C} \times_{\mathfrak{c}_1\mathcal{C}} (\mathfrak{c}_1\mathcal{C})_{\mathbb{1}/} \\ & \searrow \quad \swarrow & \\ & \mathfrak{c}_1\mathbb{P}\mathcal{C} & \end{array}$$

is obtained from

$$\begin{array}{ccc} \mathbb{P}\Omega\mathcal{C} & \longrightarrow & \mathfrak{c}_1\Omega\mathcal{C} \\ & \searrow \quad \swarrow_{=} & \\ & \mathfrak{c}_1\Omega\mathcal{C} & \end{array}$$

by left Kan extension along $\mathfrak{c}_1\Omega\mathcal{C} \to \mathfrak{c}_1\mathbb{P}\mathcal{C}$.

Inductive unfolding

- Thus we can recover $\mathcal{C}$ from the diagram

$$\begin{array}{ccc} \mathbb{P}\Omega\mathcal{C} & & \\ \downarrow & & \\ \mathfrak{c}_1\Omega\mathcal{C} & \longrightarrow & \mathfrak{c}_1\mathbb{P}\mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & \mathfrak{c}_1\mathcal{C}. \end{array}$$

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- Here $\mathbb{P}\Omega\mathcal{C}$ has duals when $\mathcal{C}$ is left 3-rigid, so can proceed by induction to recover $\mathcal{C}$ from its categorical chain complex.

The End

Thank you!

Closed s.m. enriched ∞-categories

Theorem (H.–Nikolaus, [$*$])

$\mathcal{V}$ a s.m. $(\infty, 1)$-category. There is an equivalence of $(\infty, 1)$-categories between

- closed s.m. $(\infty, 1)$-categories $\mathcal{C}$ equipped with a lax s.m. functor $\phi\colon \mathcal{C} \to \mathcal{V}$
- and **pinned** closed s.m. $\mathcal{V}$-$(\infty, 1)$-categories $\mathcal{D}$, $\mathcal{B} \to U\mathcal{D}$.

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- The equivalence is given by

$$(\mathcal{C}, \phi) \mapsto (\phi_*\overline{\mathcal{C}},\ \mathcal{C} \to U\phi_*\overline{\mathcal{C}}),$$

$$(\mathcal{D}, \psi) \mapsto (\mathcal{B} \xrightarrow{\psi} U\mathcal{D} \xrightarrow{\mathcal{D}(\mathbb{1},-)} \mathcal{V}).$$

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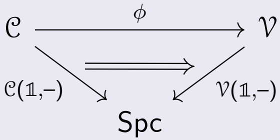

  commutes.
- and closed s.m. $\mathcal{V}$-$(\infty, 1)$-categories.

Kan extensions

Theorem (H.–Nikolaus)

Suppose $p\colon \mathcal{B} \to \mathcal{A}$ is a cocartesian fibration of symmetric monoidal $(\infty, 1)$-categories whose underlying functor is also a cartesian fibration. Then left Kan extension along $\mathcal{B}_{\mathbb{1}} \to \mathcal{B}$ induces a fully faithful functor

$$\mathrm{Mon}_{\mathcal{B}_{\mathbb{1}}^{\otimes}}(\mathrm{Cat}_{(\infty,n)}) \hookrightarrow \mathrm{Mon}_{\mathcal{B}^{\otimes}}(\mathrm{Cat}_{(\infty,n)})_{/\mathcal{A}(\mathbb{1},p(-))}$$

with image those transformations $\alpha\colon F \to \mathcal{A}(\mathbb{1}, p(-))$ such that

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$(\star)$ for $f : x \to y$ a p-cartesian morphism, the square

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ \mathcal{A}(\mathbb{1}, p(x)) & \xrightarrow[\mathcal{A}(\mathbb{1},p(f))]{} & \mathcal{A}(\mathbb{1}, p(y)) \end{array}$$

is a pullback.

Kan extensions

Apply this to $p\colon \mathfrak{c}_1\mathbb{P}\mathcal{C} \to \mathfrak{c}_1\mathcal{C}$:

Corollary

Left Kan extension along $\mathfrak{c}_1\Omega\mathcal{C} \to \mathfrak{c}_1\mathbb{P}\mathcal{C}$ induces a fully faithful functor

$$\mathrm{Mon}_{\mathfrak{c}_1\Omega\mathcal{C}}(\mathrm{Cat}_{(\infty,n)}) \hookrightarrow \mathrm{Mon}_{\mathfrak{c}_1\mathbb{P}\mathcal{C}}(\mathrm{Cat}_{(\infty,n)})_{/\mathfrak{c}_1\mathcal{C}(\mathbb{1},p(-))}$$

with image those transformations $\alpha\colon F \to \mathfrak{c}_1\mathcal{C}(\mathbb{1}, p(-))$ such that $(\star)$ holds.

For our triangle

$$\begin{array}{ccc} \mathbb{P}^2\mathcal{C} & \longrightarrow & \mathfrak{c}_1\mathbb{P}\mathcal{C} \times_{\mathfrak{c}_1\mathcal{C}} (\mathfrak{c}_1\mathcal{C})_{\mathbb{1}/} \\ & \searrow \quad \swarrow & \\ & \mathfrak{c}_1\mathbb{P}\mathcal{C}, & \end{array}$$

condition $(\star)$ is precisely equivalent to $\mathbb{P}\mathcal{C} \to \mathfrak{c}_1\mathcal{C}$ being a cocartesian fibration!