Well-pointed endofunctors on ∞-categories (Joint work with Mathieu Anel)

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Proof.

Define an ordinally indexed sequence:

$$\begin{cases} f^0(x) & = & x \\ f^{\beta+1}(x) & = & f(f^\beta(x)) \\ f^\alpha(x) & = & \sup_{\beta<\alpha} f^\beta(x) \quad \text{(If } \alpha \text{ is limit)} \end{cases}$$

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It has to stabilize at some stage, this gives the smallest fixed-point above x. □

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It axiomatizes the idea of a construction that can be iterated an ordinal number of times until it stabilizes to a “fixed-point”.

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So far, (T, t) is a **pointed endofunctor**.

Given a pointed endofunctor, we can iterate it (assuming the appropriate colimit exists):

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and this continue with $T^{\beta+1}(X) = T(T^\beta(X))$ and $T^\alpha(X) = \mathrm{Colim}_{\beta<\alpha} T^\beta(X)$ to construct a functor:

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where Ord is the poset of ordinals. Potentially this is only defined for some X and some α if $\mathcal{C}$ doesn't have all the required colimits.

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and there is in fact an infinite number of possible sequences (there are n different maps from $T^{n-1}(X)$ to $T^n(X)$ we could use).

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This condition can be written as

$$Tt = tT$$

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We call this the category $Fix(T)$ of ***fixed-points*** *of T.*

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- *$T^\alpha(X)$ exists.*
- *The colimit $T^\alpha(X) = Colim_{\beta<\alpha} T^\beta(X)$ defining $T^\alpha(X)$ is preserved by T.*

Then $T^\alpha(X)$ is the reflection of X on the category $Fix(T)$ of fixed-points of T.

All this comes from:

BULL. AUSTRAL. MATH. SOC. 18C15, 18A40, 18D10
VOL. 22 (1980), 1-83.

A UNIFIED TREATMENT OF TRANSFINITE CONSTRUCTIONS
FOR FREE ALGEBRAS, FREE MONOIDS, COLIMITS,
ASSOCIATED SHEAVES, AND SO ON

G.M. KELLY

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This allows to give "explicit" constructions of colimits of M-algebras, or of the free M-algebras.

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So starting with an M-module S, we have the colimit

$$S \xrightarrow{a} S \xrightarrow{a} S \xrightarrow{a} S \xrightarrow{a} S \to \cdots \to S[a^{-1}] = S \otimes_M M[a^{-1}]$$

Taking $S = M$ will compute $M[a^{-1}]$.

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2. So if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two (accessible) reflective subcategories of $\mathcal{C}$, with reflection $R_1 : \mathcal{C} \to \mathcal{C}_1$ and $R_2 : \mathcal{C} \to \mathcal{C}_2$, the composite $R_1 R_2$ is a well-pointed endofunctor whose iteration will produce the reflection on $\mathcal{C}_1 \cap \mathcal{C}_2$.

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In what follows ∞-categories means $(\infty, 1)$-categories. So by 2-categories, I really mean weak $(2, 1)$-categories. Everything applies to 2-categories as well - but just keep in mind that when I talk about a 2-cell I always mean an invertible one, and all colimits are pseudo-colimits.

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Is this a fixed-point? Does it provide a construction of $\mathcal{M}[a^{-1}]$?

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Theorem (Voevodsky 1998 for 1-categories, Robalo 2015 for ∞-categories)

Given a symmetric monoidal (∞-)category $\mathcal{M}$ and $a \in \mathcal{M}$ an object, the pseudo-colimit

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Theorem (A version of the Group completion theorem)

If $\mathcal{M}$ is a E_∞-monoid in the ∞-category of spaces and $a \in \mathcal{M}$, the homotopy colimit of

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[a] G is perfect if the commutator subgroup $[G, G]$ is G.

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A **braided endofunctor** on an ∞-category $\mathcal{C}$ is triple (T, t, τ) where (T, t) is a pointed endofunctor on $\mathcal{C}$ and τ is a (invertible) 2-cell

$$\tau : T \otimes t \to t \otimes T$$

in $\mathrm{End}(\mathcal{C})$.

Definition

A **pointed endofunctor** on an ∞-category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$ together with a natural transformation $t : \mathrm{Id} \to T$.

A **braided endofunctor** on an ∞-category $\mathcal{C}$ is triple (T, t, τ) where (T, t) is a pointed endofunctor on $\mathcal{C}$ and τ is a (invertible) 2-cell

$$\tau : T \otimes t \to t \otimes T$$

in $\mathrm{End}(\mathcal{C})$.

Remark

If (T, t, τ) is a braided endofunctor on an ∞-category $\mathcal{C}$, then (T, t) is a well-pointed endofunctor on the homotopy category $Ho(\mathcal{C})$.

Theorem (A., H.)

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Let $\mathcal{C}$ be an ∞-category with a braided endofunctor (T, t, τ) then the following ∞-categories are equivalent:

- *The full subcategory of $\mathcal{C}$ of objects X such that t_X is an isomorphism.*
- *The ∞-category of T-algebras (for the pointed endofunctor T).*
- *The full subcategory of objects of $\mathcal{C}$ that are orthogonal to t_Y for all y.*

These equivalent categories are denoted by $Fix(T)$.

The proof is mostly the same as for 1-categories.

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The proof is mostly the same as for 1-categories. In fact most of it happens in the homotopy category, and we don't even need τ to be natural.

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X & \xrightarrow{t} & T(X) & \xrightarrow{tT} & T^2(X) & \xrightarrow{tT^2} & \dots & \longrightarrow & Y \\
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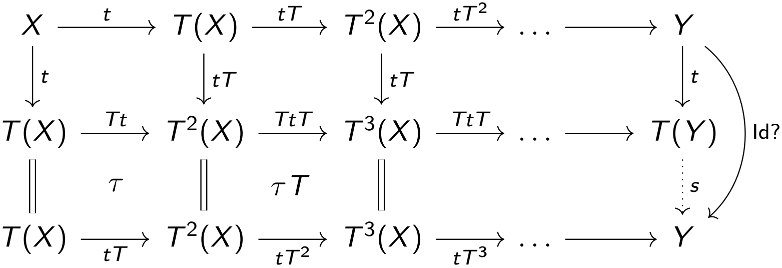

It works in 1-category theory because shifting the diagram preserve the colimit:

$$\begin{array}{ccccccccc}
X & \xrightarrow{t} & TX & \xrightarrow{tT} & T^2X & \xrightarrow{tT^2} & \dots & \longrightarrow & Y \\
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and this diagram is the same as the outer part of the previous one:

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But for this to work in ∞-category theory (or 2-category theory), we also need the 2-cells of these two diagram to be the same: Changing the 2-cell making the square commute can affect the morphism between the colimits!

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That is we need that for each object X:

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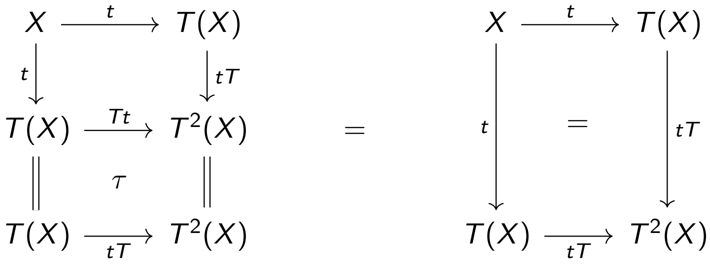

We call $\tau^{(2)}(X)$ the 2-cell inside the left rectangle.

Importantly, $\tau_X^{(2)}$ is an endomorphism 2-cell, that is its source and target are the same 1-cells.

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Let (T, t, τ) be a braided endofunctor on an ∞-category. Let X be an object and $\alpha > 0$ a limit ordinal such that:

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So to ensure convergence with the above theorem and compute $\mathcal{M}[a^{-1}]$, we need to know that the isomorphism

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So to ensure convergence with the above theorem and compute $\mathcal{M}[a^{-1}]$, we need to know that the isomorphism

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is trivial. This is a stronger requirement than the 3-cycle condition we mentioned earlier.

We can do better. Let's go back to our diagram:

$$\begin{array}{ccccccccc}
X & \xrightarrow{t} & T(X) & \xrightarrow{tT} & T^2(X) & \xrightarrow{tT^2} & \cdots & \longrightarrow & Y \\
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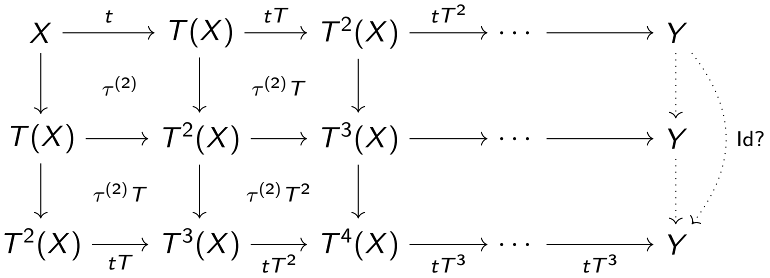

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\downarrow & \left(\tau^{(2)}\right)^{-1} T & \downarrow & \left(\tau^{(2)}\right)^{-1} T^2 & \downarrow & & & & \vdots \\
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\end{array}$$

So to ensure convergence, we can replace $\tau^{(2)}$ by:

$$
\tau^{(3)}(X) = \begin{array}{ccc}
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T(X) & \longrightarrow & T^2(X) \\
{\scriptstyle tT}\big\downarrow & (\tau^{(2)})^{-1}T & \big\downarrow{\scriptstyle tT^2} \\
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In the special case of multiplication by an element in a symmetric monoid (or by an object in a symmetric monoidal category) that we discussed earlier, $\tau^{(3)}$ is the 3-cycle

$$a \otimes a \otimes a \to a \otimes a \otimes a$$

Theorem (A.,H.)

Let (T, t, τ) be a braided endofunctor on an ∞-category. Let X be an object and α a limit ordinal such that:

1. $T^{\alpha}(X)$ *exists.*
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3. ~~$\tau^{(2)}(T^{\alpha}(X)) \sim Id$.~~ $\tau^{(3)}(T^{\alpha}(X)) \sim Id$.

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Moreover, the point of using this $(\tau^{(2)})^{-1}$ in the definition of $\tau^{(3)}$, is so that we have:

Proposition (A. , H.)

If Y is a fixed-point of T, then $\tau^{(3)}(Y) \sim Id$. In particular, the last condition is necessary in the previous theorem.

Definition (A.,H.)

We say that a Braided endofunctor (T, t, τ) is

1. **Strongly well-pointed** if $\tau_X^{(2)} \sim \text{Id}$ for all X.
2. **Well-pointed** if $\tau_X^{(3)} \sim \text{Id}$ for all X.
3. **Eventually well-pointed** if for each object X, $\tau_{T^\alpha(X)}^{(3)} \sim \text{Id}$ for α large enough.

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Moreover, if S and T are (strongly) well-pointed then ST is (strongly) well-pointed.

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Let $\mathcal{B}$ be the free monoidal ∞-category generated by a "braided object".

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2. together with an arrow $t : 1 \to T$,
3. a 2-cell $\tau : T \otimes t \simeq t \otimes T \in \mathrm{Hom}(T, T^2)$.

The condition on $\tau^{(3)}$ still looks mysterious however... But in order to establish our result we did made some progress toward understanding it better.

In order to better understand the role of the various map (and higher arrows) we can build by combining t and τ let's consider:

Definition

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That is, a braided endofunctor on $\mathcal{C}$ is the same as a monoidal functor $\mathcal{B} \to \mathrm{End}(\mathcal{C})$, i.e. an action of $\mathcal{B}$ on $\mathcal{C}$.

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So $\mathcal{B}$ is a 2-category, and the 2-cells correspond to braids.

We can be more precise on what the composition and monoidal structure on $\mathcal{B}$ actually is:

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$\mathcal{B}$ is equivalent to a strictly monoidal strict 2-category, in which all 2-arrows have inverses.

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3. 2-morphism $f \Rightarrow g$ are braids on $m - n$ strand connecting the $m - n$ points not in the image of f to the $m - n$ points not in the image of g.

We draw 1-morphism by putting circle around the element in the image, for example

$$1 \quad \textcircled{2} \quad 3$$

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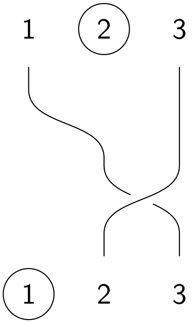
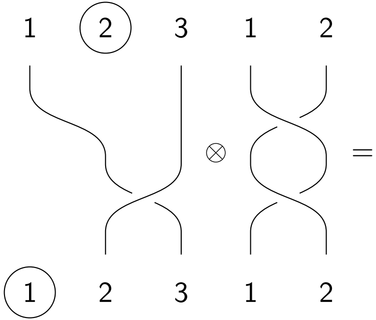
2-morphisms are braids connecting the non-circled elements:



The tensor product is horizontal concatenation:



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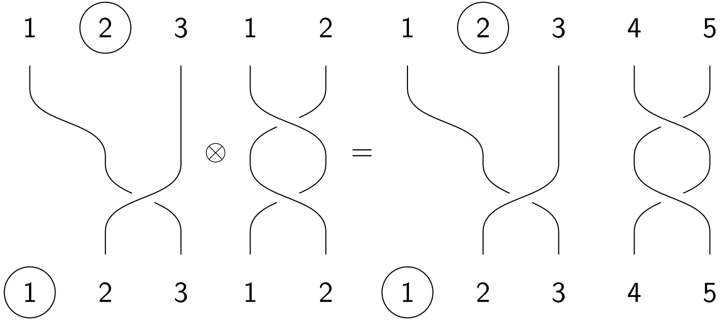

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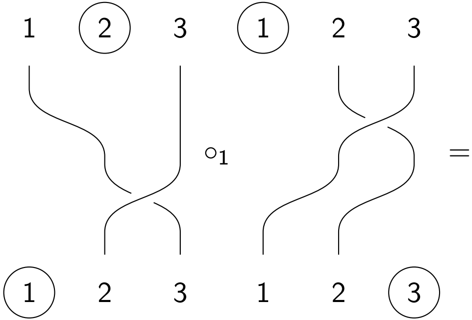

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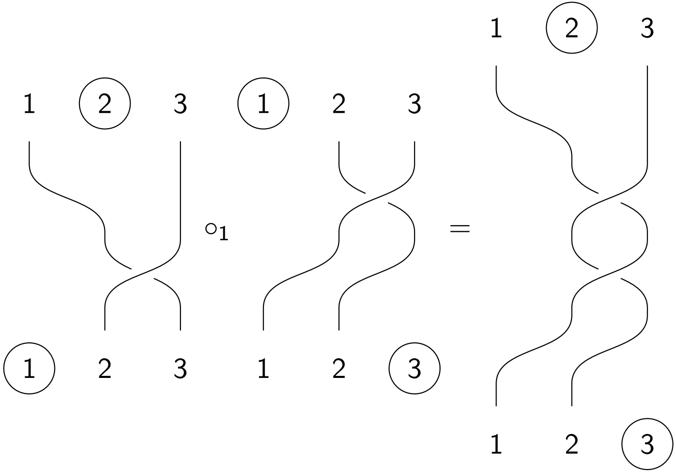

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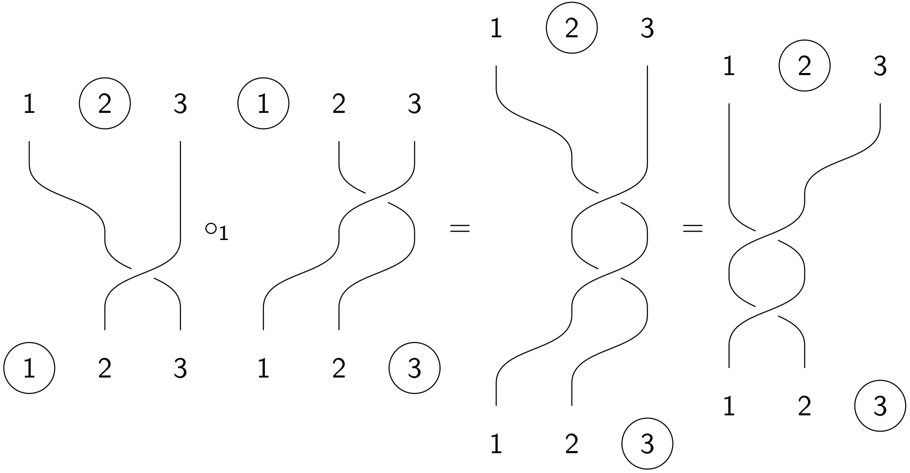

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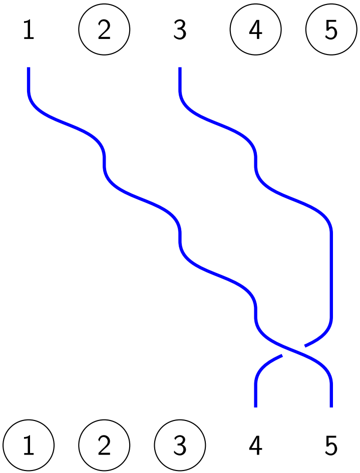

$\circ_0 \qquad =$

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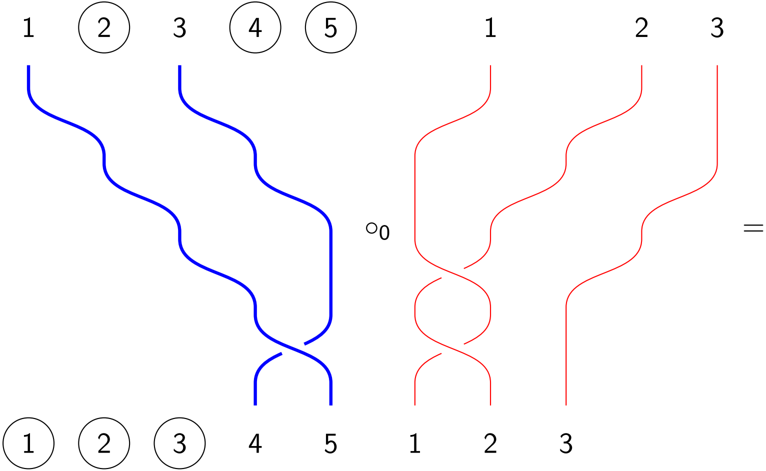

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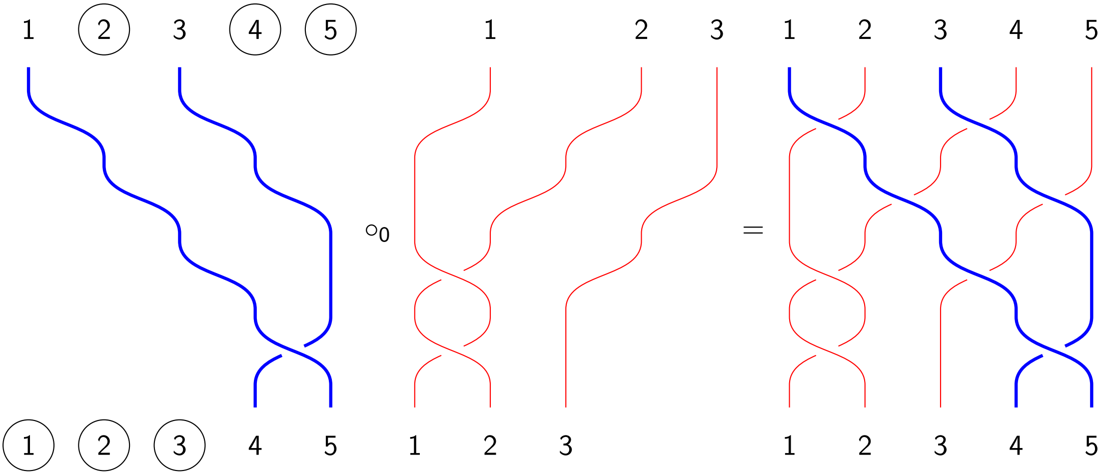

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$$\mathrm{Id}_T = \textcircled{1} \qquad t = 1$$

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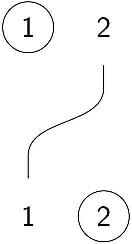
And:

$$\tau : Tt \to tT =$$



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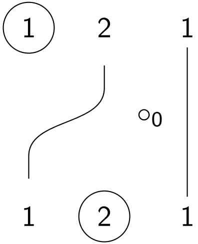

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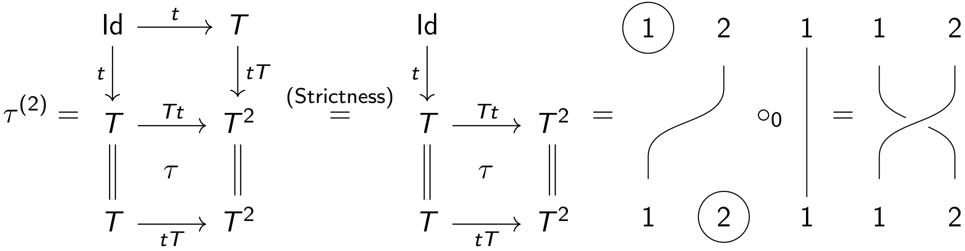

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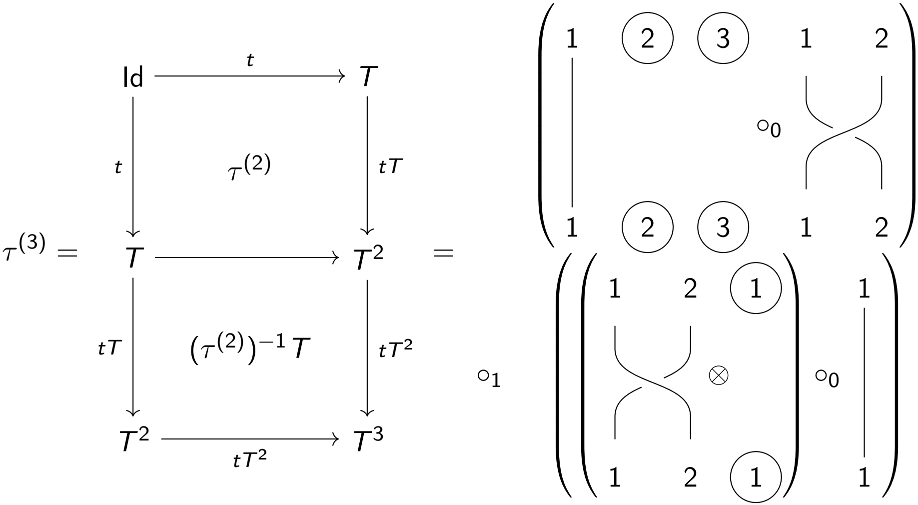

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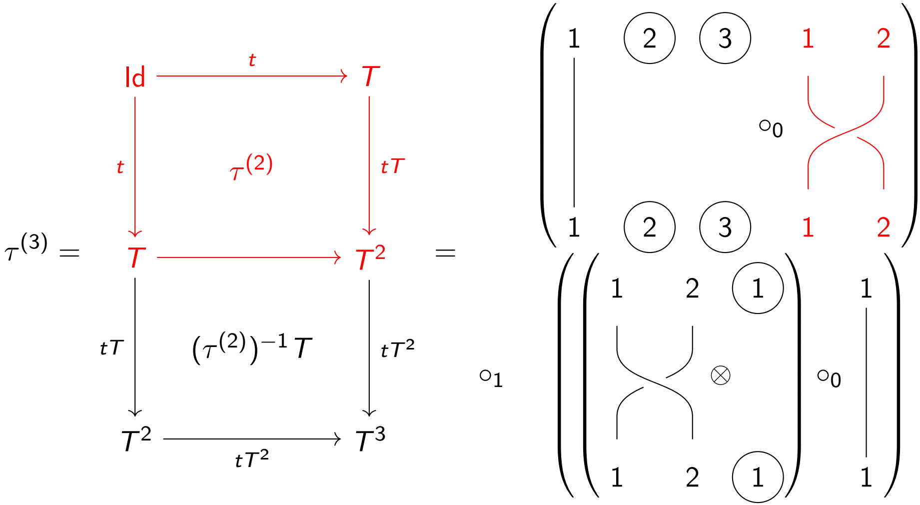

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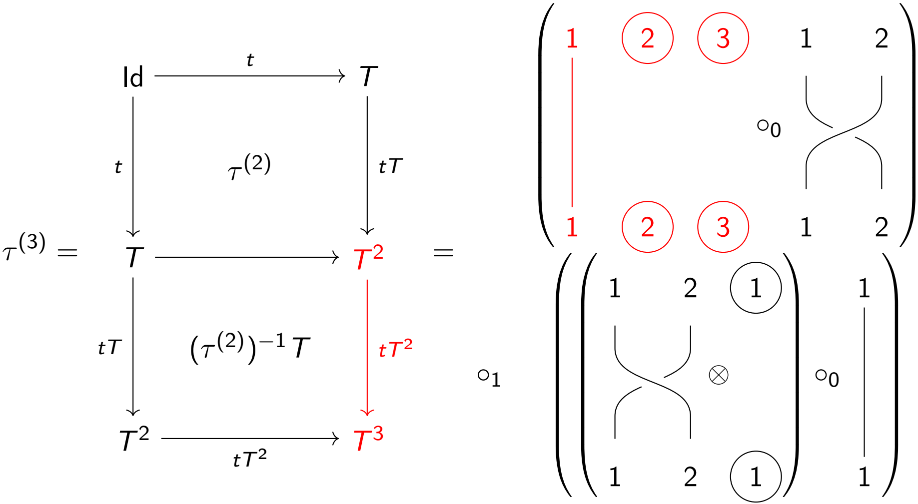

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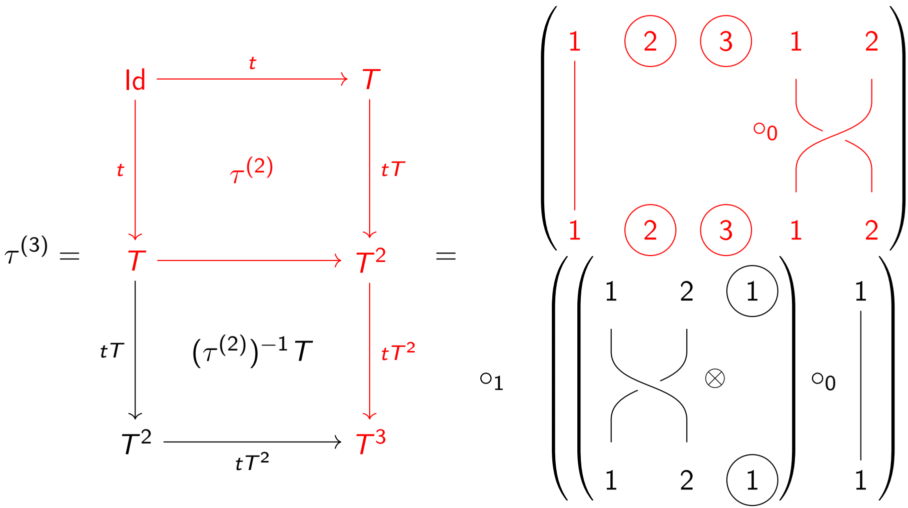

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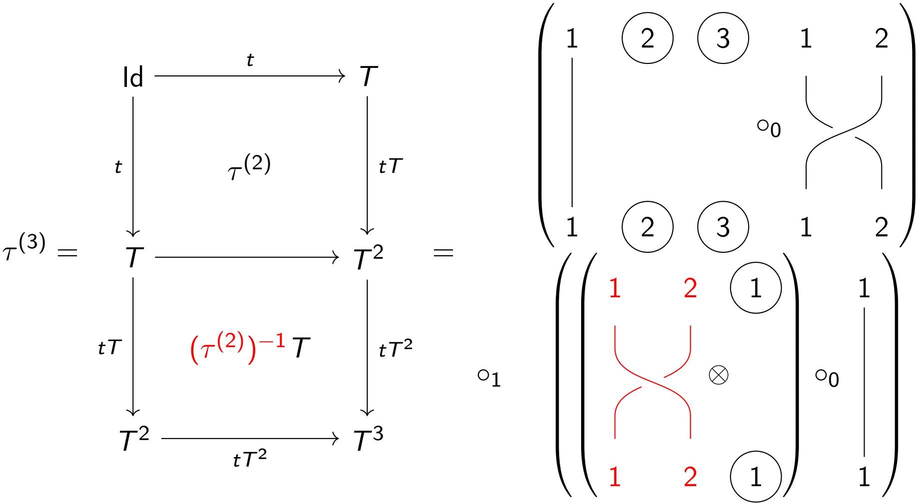
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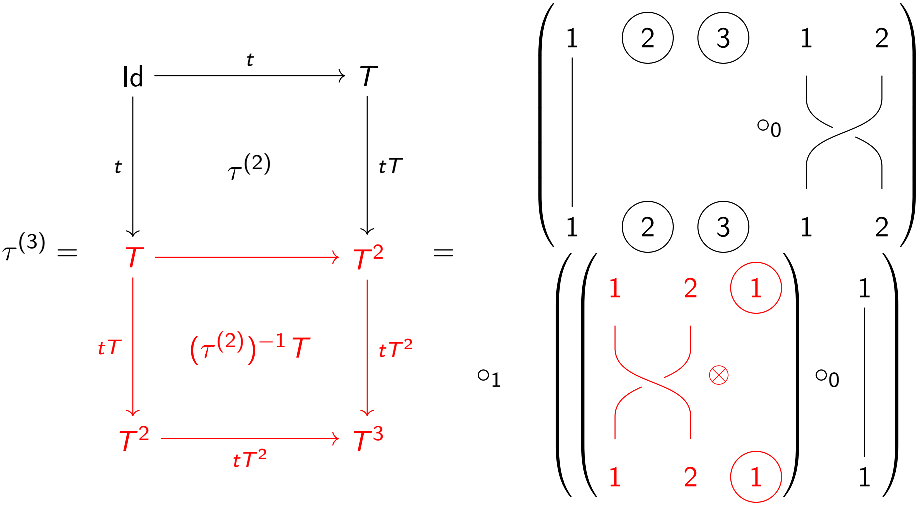

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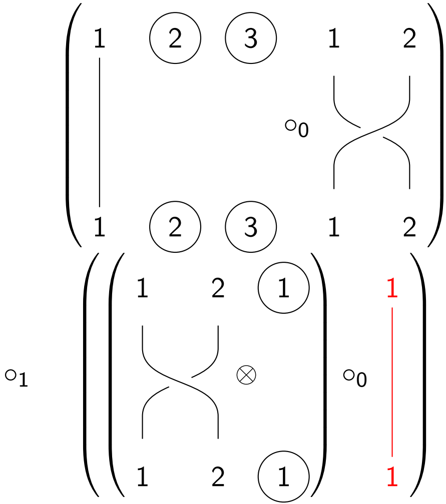

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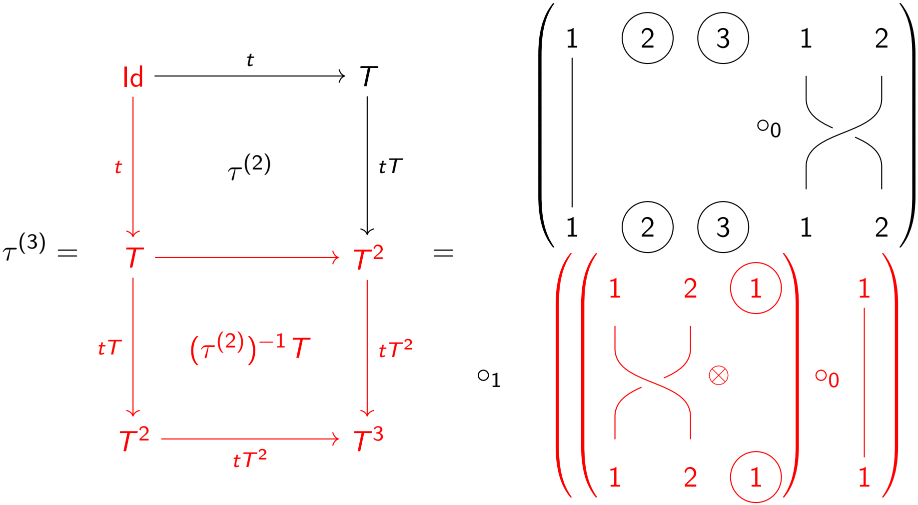

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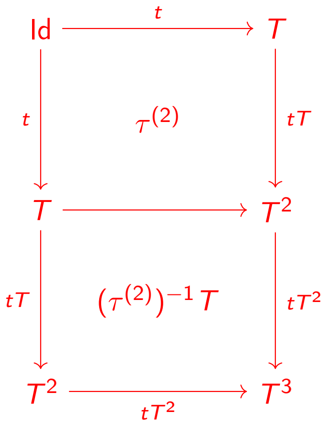

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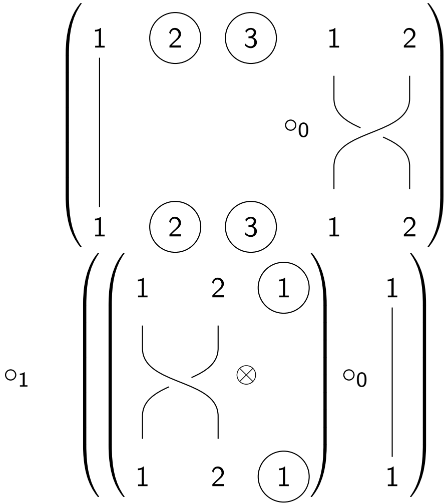

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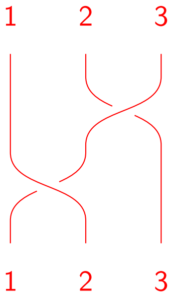

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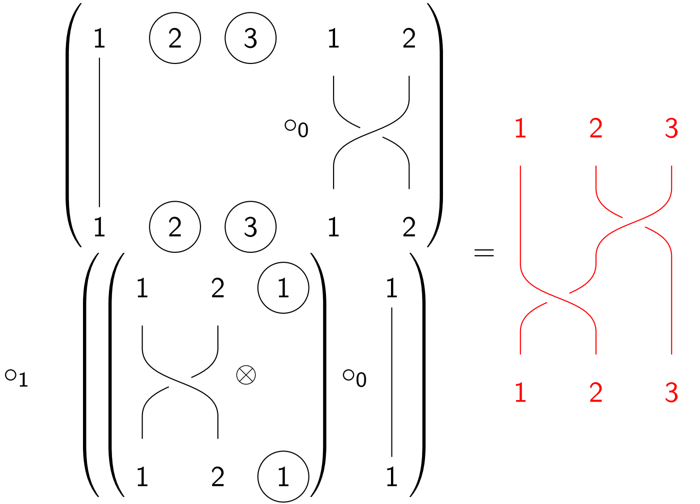

Is a 3-cycle realized as an element of degree 0.

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In particular:

Remark

1. *T is **strongly well-pointed** iff all the $B_n \to \pi_1(Hom(X, T^n X))$ induced by T are trivial.*
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Proposition (A.,H.)

If in $\mathcal{C}$, all the $\pi_1(Hom(X,Y),\bullet)$ are abelian groups, then every braided endofunctor on $\mathcal{C}$ is well-pointed (i.e. $\tau^{(3)} \sim 1$).

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If T is a braided endofunctor on $\mathcal{C}$ which has colimits, there is a quotient $T^{\omega} \to S$ so that S is strongly well-pointed and has the same fixed-point as T.

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Remark

Here we really mean that $Hom(S(X),Y)$ is a subspace of $Hom(T^\omega(X),Y)$, not elements of $Hom(T^\omega(X),Y)$ equipped with an additional homotopies trivializing these cells.

Theorem (A.,H.)

If T is a braided endofunctor on $\mathcal{C}$ which has colimits, there is a quotient $T^\omega \to S$ so that S is strongly well-pointed and has the same fixed-point as T. S is defined by:

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The fact that this sort of thing is possible is closely related to Quillen's $+$-construction and the fact that the Kernel of the degree map is a perfect group (for $n \geqslant 5$).

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However, this last quotient can be quite complicated...

Conjecture (Structure theorem for strongly well-pointed endofunctors)

*The poset $\mathbb{N}$, equipped with the addition as a monoidal structure, is equivalent to the free monoidal ∞-category generated by **strongly well-pointed object**,*

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- *An object T.*
- *A map $t : 1 \to T$.*
- *A "braiding" 2-cell $\tau : T \otimes t \to t \otimes T$.*
- *A 3-cell $\Theta : \tau^{(2)} \simeq Id$ witnessing that the previous braided object is strongly well-pointed.*

Thank you!