Notions of Cauchy (co)completeness for normed categories

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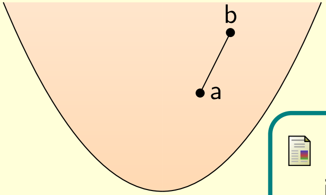
[1]Based on joint work with Maria Manuel Clementino and Walter Tholen.

Motivation: the Bear and Weiss metric

Example. Let K be a convex subset of a vector space and $a, b \in K$ with $a \neq b$:

$d(a, b) =$ how far is the ray from b via a in K?



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Bear, Herbert S. and Weiss, Max L. (1967). "An intrinsic metric for parts". In: *Proceedings of the American Mathematical Society* **18**.(5), pp. 812–817.

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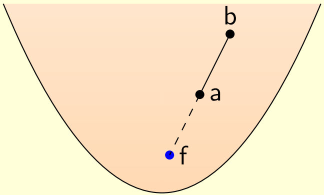

$$a = (1 - \alpha)f + \alpha b, \quad \alpha \in]0, 1[.$$

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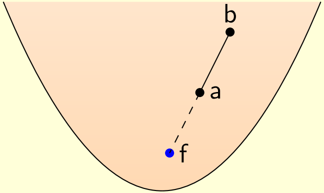
$$d(a,b) = \inf\{-\log(\alpha_f) \mid f \in K, a \text{ in }]f,b[\}.$$



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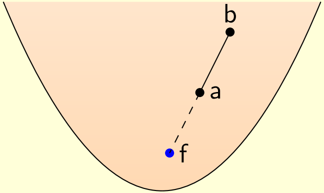

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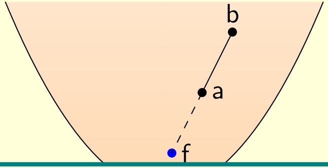

Definition. A **normed category** $\mathbb{X}$ is an ordinary category with (small) normed hom-sets

$$|-|\colon \mathbb{X}(x,y) \longrightarrow [0,\infty]$$

satisfying $0 \geq |1_x|$ and $|g| + |f| \geq |g \cdot f|$.

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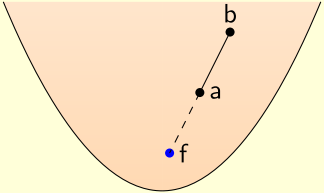

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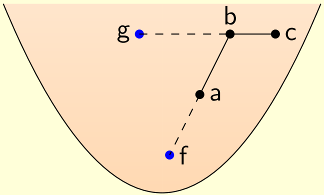

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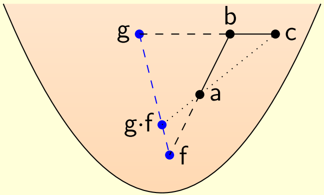

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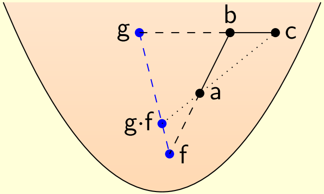

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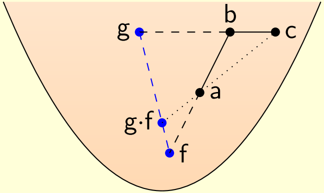

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Definition. For a quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$.

- A **$\mathcal{V}$-normed** set is given by $|-|\colon A \to \mathcal{V}$.
- A **$\mathcal{V}$-normed map** $(A, |-|) \to (B, |-|)$ is a map $f\colon A \to B$ satisfying

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B. \\ & \leq & \\ {\scriptstyle |-|} \searrow & & \swarrow {\scriptstyle |-|} \\ & \mathcal{V} & \end{array}$$

This defines the category $\mathbf{Set}/\!\!/\mathcal{V}$.

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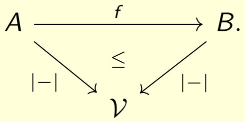

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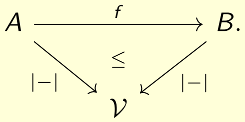

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Remark. The internal hom $[A, B]$ has carrier set $\mathbf{Set}(A, B)$ (all mappings $\varphi\colon A \to B$) with

$$|\varphi| = \bigwedge_{a \in A} [|a|, |\varphi a|].$$

Normed categories

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- whose objects are $\mathcal{V}$-normed sets,
- but whose normed hom-sets of morphisms $A \to B$ are given by the internal hom $[A, B]$ of $\mathbf{Set}/\!\!/\mathcal{V}$, that is, by all **Set**-maps $A \to B$ with
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A **$\mathcal{V}$-normed functor** $F\colon \mathbb{X} \to \mathbb{Y}$ is a $\mathbf{Set}/\!\!/\mathcal{V}$-functor: each $F\colon \mathbb{X}(x,x') \to \mathbb{Y}(Fx,Fx')$ is in $\mathbf{Set}/\!\!/\mathcal{V}$, that is $|f| \leq |Ff|$.

A **$\mathcal{V}$-normed natural transformation** $\alpha\colon T \to S$ is a $\mathbf{Set}/\!\!/\mathcal{V}$-natural transformation: a family $(\alpha_x\colon E \to \mathbb{Y}(Tx,Sx))_x$ ... with $k \leq |\alpha_x|$.

We simpy write $\mathbf{Cat}/\!\!/\mathcal{V}$ and $\mathbf{CAT}/\!\!/\mathcal{V}$ instead of $(\mathbf{Set}/\!\!/\mathcal{V})$-$\mathbf{Cat}$ and $(\mathbf{Set}/\!\!/\mathcal{V})$-$\mathbf{CAT}$, respectively.

For every closed symmetric monoidal category $\mathcal{W}$,

$$[-,-]\colon \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{W}$$

makes $\mathcal{W}$ a $\mathcal{W}$-category.

In particular, $\mathbf{Set}/\!\!/\mathcal{V}$ becomes a $\mathcal{V}$-normed category

- whose objects are $\mathcal{V}$-normed sets,
- but whose normed hom-sets of morphisms $A \to B$ are given by the internal hom $[A,B]$ of $\mathbf{Set}/\!\!/\mathcal{V}$, that is, by all $\mathbf{Set}$-maps $A \to B$ with
$$|\varphi| = \bigwedge_{a\in A} [|a|, |\varphi a|].$$

We write $\mathbf{Set}\|\mathcal{V}$ to denote this $\mathcal{V}$-normed category, then

$$(\mathbf{Set}\|\mathcal{V})_\circ = \mathbf{Set}/\!\!/\mathcal{V}.$$

Change of base

Remark. The functor $s\colon \mathbf{Set}/\!\!/\mathcal{V} \to \mathcal{V}$

$$f\colon A \to B \longmapsto \bigvee_{a\in A} |a| \le \bigvee_{b\in B} |b|$$

is symmetric strict monoidal and induces

$$s\colon \mathbf{Cat}/\!\!/\mathcal{V} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat},$$

$$\mathbb{X} \longmapsto (\mathrm{Ob}\,\mathbb{X}, s\mathbb{X}(x,y) = \bigvee_{f\colon x\to y} |f|).$$

Reminder: The 2-category $\mathcal{V}$-**Cat** is given by the following data:

- A **$\mathcal{V}$-category** X consists of a set X and a function $X(-,-)\colon X \times X \to \mathcal{V}$ satisfying

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$$i\colon \mathcal{V} \longrightarrow \mathbf{Set}/\!\!/\mathcal{V}, \quad v \longmapsto (\{\star\}, |\star| = v)$$

which is symmetric strong monoidal and induces the functor (right adjoint to s)

$$i\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}/\!\!/\mathcal{V},$$

$$X \longmapsto \mathbb{X} \text{ “indiscrete”}, |(x,y)| = X(x,y).$$

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Remark. The internal hom $[\mathbb{X}, \mathbb{Y}]$ is given by the $\mathcal{V}$-normed functors $\mathbb{X} \to \mathbb{Y}$ and all natural transformations between them, normed by

$$|\alpha| = \bigwedge\{|\alpha_x| \mid x \in \mathrm{Ob}\,\mathbb{X}\}.$$

Normed convergence

Definition. Let $s = (x_m \xrightarrow{s_{m,n}} x_n)_{n \geq m \in \mathbb{N}}$ be a sequence in the $\mathcal{V}$-normed category $\mathbb{X}$.

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1. x is a colimit of s in the ordinary category $\mathbb{X}$, with a colimit cocone $(x_n \xrightarrow{\gamma_n} x)$ so that
2. for all objects y in $\mathbb{X}$, the canonical $\mathbf{Set}$-bijection

$$\mathrm{Nat}(s, \Delta y) \longrightarrow \mathbb{X}(x, y)$$

is an isomorphism in $\mathbf{Set}/\!\!/\mathcal{V}$, that is

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Clementino, Maria Manuel, Hofmann, Dirk, and Tholen, Walter (2025). "Cauchy convergence in V-normed categories". In: *Advances in Mathematics* **470**, p. 110247.

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Remark. Condition 2 splits in two conditions:

2a. $\bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} |\gamma_n| \geq k$.

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Proposition. Normed colimits are unique up to k-isomorphism.

Proposition. Every left adjoint normed functor $F \colon \mathbb{X} \to \mathbb{Y}$ preserves normed colimits.

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An expected definition:

Definition. For a $\mathcal{V}$-normed category $\mathbb{X}$, we say that

- a sequence $s = (x_m \xrightarrow{s_{m,n}} x_n)_{n \geq m \in \mathbb{N}}$ in $\mathbb{X}$ is **Cauchy** if

$$k \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq m \geq N} |s_{m,n}|,$$

- and $\mathbb{X}$ is **Cauchy cocomplete** if every Cauchy sequence in $\mathbb{X}$ has a normed colimit in $\mathbb{X}$.

Example: Normed vector spaces

Definition. A **semi-norm** $\|x\|: X \to [0, \infty]$ on a (real) vector space X satisfies:

- $\|0\| = 0$.
- $\|a \cdot x\| = |a| \cdot \|x\| \quad (a \in \mathbb{R},\ a \neq 0)$.
- $\|x + y\| \leq \|x\| + \|y\|$.

A semi-norm is a **norm** whenever also

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Remark. For a sequence $s = (a_n)_n$ in X:

$$\star \xrightarrow{a_0} \star \xrightarrow{a_1} \star \cdots\cdots\to$$

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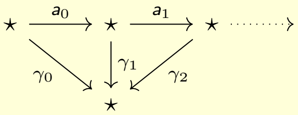

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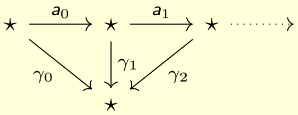

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- $\gamma_1 = \gamma_0 - a_0$, $\gamma_2 = \gamma_0 - (a_0 + a_1)$, ...
- $(\gamma_n)_n$ is normed colimit of s iff $\gamma_0 = \sum_{i=0}^{\infty} a_i$.

Example. A sequence $s = (x_n)$ in a $\mathcal{V}$-category X is **forward Cauchy** whenever

$$k \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq m \geq N} X(x_m, x_n).$$

An element $x \in X$ is a **forward limit** of s if

$$X(x, y) = \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq N} X(x_n, y),$$

for all $y \in X$, and X is (forward) complete whenever every forward Cauchy sequence converges.

Comparison with the quantale case

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Remark. For a $\mathcal{V}$-category X:

$$X \text{ complete} \iff i(X) \text{ Cauchy cocomplete.}$$

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Remark. For a $\mathcal{V}$-category X:

$$X \text{ complete} \iff i(X) \text{ Cauchy cocomplete.}$$

Remark. Via the normed functor

$$i(\mathcal{V}) \longrightarrow \mathbf{Set}\|\mathcal{V}, \quad v \longmapsto (\{\star\}, |\star| = v),$$

$$[(\star, u), (\star, v)] = \mathcal{V}(u, v),$$

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Example. A sequence $s = (x_n)$ in a $\mathcal{V}$-category X is **forward Cauchy** whenever

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An element $x \in X$ is a **forward limit** of s if

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Remark. Under certain conditions on $\mathcal{V}$:

- Presheaf categories are Cauchy cocomplete.
- Normed colimits are weighted colimits.
- Cauchy cocompleteness is stable for internal homs, products, strict equifiers, ...

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Representable adjunction: $X(x,-) \dashv X(-,x)$.

Reminder:

- $\mathcal{V}$-distributor $\varphi\colon X \nrightarrow Y = \varphi\colon X^{\mathrm{op}} \otimes Y \to \mathcal{V}$.
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Reminder: Isbell conjugation adjunction:

$$[X,\mathcal{V}]^{\mathrm{op}} \underset{(-)^\vee}{\overset{(-)^\vee}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} [X^{\mathrm{op}},\mathcal{V}]$$

$$\varphi\colon E \nrightarrow X \text{ left adjoint} \implies \varphi \dashv \varphi^\vee$$

$$\varphi\colon E \nrightarrow X \text{ left adjoint} \implies \varphi \cong \varphi^{\vee\vee}$$

For more information, see (for instance)

Avery, Tom and Leinster, Tom (2021). "Isbell conjugacy and the reflexive completion". In: *Theory and Applications of Categories* **36**.(12), pp. 306–347.

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Borceux, Francis (1994). *Handbook of categorical algebra 1. Basic category theory.* Vol. 50. Encyclopedia of Mathematics and its Applications. Cambridge University Press, pp. xvi + 345.

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Lemma. For natural transformations

$$\alpha\colon \mathbf{A}(a,-) \longrightarrow \varphi \quad \text{and} \quad \beta\colon \varphi \longrightarrow \mathbf{A}(a,-).$$

Then

$$\varphi \dashv \varphi^\vee \text{ with unit } \eta = [(v,u)] \iff \alpha\beta = 1_\Phi.$$

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Theorem. $\varphi\colon \mathbf{E} \nrightarrow \mathbf{A}$ is left adjoint if and only if φ is a split retract of a representable.

Theorem. A category $\mathbf{A}$ is Lawvere complete if and only if $\mathbf{A}$ is idempotent complete.

The normed case

Lemma. Let $\varphi\colon \mathbb{E} \nrightarrow \mathbb{A}$ in $\mathbf{Dist}/\!\!/\mathcal{V}$ and

$$\alpha\colon \mathbb{A}(a,-) \longrightarrow \varphi \quad \text{and} \quad \beta\colon \varphi \longrightarrow \mathbb{A}(a,-)$$

be (ordinary) natural transformations, here

- α corresponds to $u \in \varphi(a)$ and
- $\beta^\vee\colon \mathbf{A}(-,a) \to \varphi^\vee$ to $v = \beta \in \varphi^\vee(a)$.

Then the following assertions are equivalent.

(i) $\varphi \dashv \varphi^\vee$ in $\mathbf{Dist}/\!\!/\mathcal{V}$ with unit $\eta = [(v,u)]$ where $k \leq |u|$ and $k \leq |v|$.

(ii) α, β are $\mathcal{V}$-normed & $\alpha\beta = 1_\varphi$ in $\mathbf{Dist}/\!\!/\mathcal{V}$.

Reminder: Normed coends can be calculated "as in $\mathbf{Set}$". In particular, for $\varphi\colon \mathbb{E} \nrightarrow \mathbb{A} \dashv \psi\colon \mathbb{A} \nrightarrow \mathbb{E}$:

$$1 \xrightarrow{\ \eta\ } \int^x (\psi(x) \otimes \varphi(x)) \qquad \eta(\star) = [(v,u)],\ k \leq |[v,u]|$$

$$\uparrow \text{final}$$

$$\psi(x) \otimes \varphi(x) \qquad |(u,v)| = |u| \otimes |v|$$

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Example. For $\mathcal{V} = 1$ and for $\mathcal{V} = 2$, a $\mathcal{V}$-normed small category $\mathbb{A}$ is Lawvere complete if and only if $\mathbb{A}_\circ$ is idempotent complete.

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Remark. $\mathbb{X}$ Cauchy cocomplete $\implies$ $\mathbb{X}_\circ$ idempotent complete.

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