Topoi of automata

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Why topoi **of** automata? (cf. [Iwaniack 2024])

1. Automata theory is multi-faceted just like topos theory

Logic	first-order, higer (mainly second-order) logic,
Topology	Stone duality and **profinite space** of profinite words,
Algebra	**Monoids**, and Grothendieck (semi-)Galois theory [Uramoto 2025],
Category	Coalgebra, fibration, factorization systems, ...

2. Several categories of automata are topoi.

We fix a finite set Σ and consider its free monoid Σ^*.

- The presheaf topos $\mathbf{PSh}(\Sigma^*)$
- the category of $2x^\Sigma$-coalgebras $\mathbf{Coalg}(2x^\Sigma)$
- the category of orbit-finite $2x^\Sigma$-coalgebras, whose terminal object consists of regular languages!

3. Ambition for solving problems with new geometric invariants

The associated topos for a language class

- We fix a finite set Σ and consider its free monoid Σ^*.
- A language L is a subset of Σ^*.
- A **language class** $C \subset \mathcal{P}(\Sigma^*)$ is a set of languages.

We want to know language classes.

Construction [Hora n.d.]

For any language class C, we can construct **the associated Grothendieck topos** $\mathcal{T}(C)$.

- For any $C \subset \mathcal{P}(\Sigma^*)$, there is a canonical hyperconnected geometric morphism $\mathbf{PSh}(\Sigma^*) \twoheadrightarrow \mathcal{T}(C)$. (cf. [Rogers 2023])

$\mathcal{T}(C)$ subsumes the syntactic monoid

Theorem (Hora n.d.)

For any regular language L, we have

$$\mathcal{T}(\{L\}) \simeq \mathbf{PSh}(M_L),$$

where M_L is the syntactic monoid of L.

Remark (What connects topos theory and word combinatorics?)

The colimit of all monomorphisms in $\mathcal{E} = \mathbf{PSh}(\Sigma^*)$ is

$$\mathrm{colim}(\mathcal{E}_{\mathrm{mono}} \rightarrowtail \mathcal{E}) = \{\sim : \text{equiv. rel. on } \Sigma^* \mid u \sim v \implies uw \sim vw\}.$$

(cf. local state classifier in [Hora 2024a])

$\mathcal{T}(\mathrm{Reg}_\Sigma)$ captures four definitions of regular languages

The language class of **regular languages** $\mathrm{Reg}_\Sigma \subset \mathcal{P}(\Sigma^*)$ is defined in many ways.

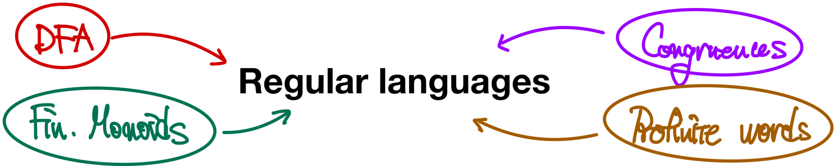
We can enrich the (conditional) equivalence between four definitions to the (structural) equivalence between four topoi!

$\mathcal{T}(\mathrm{Reg}_\Sigma)$ captures four definitions of regular languages

Theorem (Hora 2024b)

All of the following (boolean ringed) topoi are equivalent to $\mathcal{T}(\mathrm{Reg}_\Sigma)$.

- *The sheaf topos over the (boolean ringed) site of* **DFA**.
- *The sheaf topos over the (boolean ringed) site of* **finite (Σ-) monoids.**
- *The hyperconnected quotient topos of* Σ-**Set** *corresponding to finite orbit right congruences. (***finite Nerode congruence***)*
- *The continuous action topos of the profinite monoid of* **profinite words**.



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