

Internal categories, algebraic model structures and type theory

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CT, July 18 2025



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An example of some kind of (small) (2, 1)-pretopos should be satisfied by **Gpd**

... and **Gpd**($\mathcal{E}$) for suitable $\mathcal{E}$

Outline

- 1 2-categorical axioms
- 2 Algebraic homotopy theory
- 3 Type theory
- 4 Future work

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Internal categories

Let $\mathcal{E}$ be a category with pullbacks.

Definition

A category internal to $\mathcal{E}$ is:

$$\begin{array}{ccccc} & & \xrightarrow{p_1} & & \xrightarrow{d_1} \\ C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 & \xleftarrow{i} & C_0 \\ & & \xrightarrow{p_2} & & \xrightarrow{d_0} \end{array}$$

These are the objects of a 2-category $\mathbf{Cat}(\mathcal{E})$.

We denote the $(2, 1)$ -category of internal groupoids by $\mathbf{Gpd}(\mathcal{E})$.

Note that for $\mathbb{C} \in \mathbf{Cat}(\mathcal{E})$, the 2-colimit of

$$\begin{array}{ccccc} & & \xrightarrow{p_1} & & \xrightarrow{d_1} \\ C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 & \xleftarrow{i} & C_0 \\ & \xrightarrow{p_2} & & \xrightarrow{d_0} & \end{array}$$

is $\mathbb{C}$.

Cateads

Definition (Bourne-Penon, Bourke)

For a 2-category $\mathcal{K}$, a *catead* is

$$\begin{array}{ccccc} & & \xrightarrow{p_1} & & \xrightarrow{d_1} \\ C_1 \times_{C_0} C_1 & \xrightarrow{m} & C_1 & \xleftarrow{i} & C_0 \\ & & \xrightarrow{p_2} & & \xrightarrow{d_0} \end{array}$$

such that (d_1, d_0) forms a 2-sided discrete fibration.

We call its 2-colimit a *codescent object*.

So for $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$, every object is a codescent object of a *discrete* catead.

Bourke's exactness axioms

Let $\mathcal{K}$ be a 2-category.

Bourke's axioms:

- 1 $\mathcal{K}$ has pullbacks and powers by 2.
- 2 $\mathcal{K}$ has codescent objects of cateads.
- 3 Codescent morphisms are effective in $\mathcal{K}$.
- 4 Discrete objects in $\mathcal{K}$ are projective.
- 5 $\mathcal{K}$ has enough projectives.

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$Y \in \mathcal{K}$ is discrete if

$$\begin{array}{ccc} \mathbb{X} & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \phi \\ \xrightarrow{g} \end{array} & Y \end{array} \implies f = g \text{ and } \phi = 1_f$$

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$$P \twoheadrightarrow \mathbb{Y}$$

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Theorem (Carboni-Vitale)

An exact 1-category is an exact completion if and only if it has enough projectives. In this case, it is the exact completion of its projective objects.

Bourke's characterisation of $\mathbf{Cat}(\mathcal{E})$

Theorem (Bourke)

$\mathcal{E}$ has pullbacks $\implies \mathbf{Cat}(\mathcal{E})$ satisfies Bourke's axioms.

Conversely, if $\mathcal{K}$ satisfies Bourke's axioms
 $\implies \mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ where $\mathcal{E} := \mathbf{Disc}(\mathcal{K})$.

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Theorem (Bourke-Garner)

$\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ is a kind of 2-exact completion.

Refining this

Theorem (H.)

Let $\mathcal{E}$ be a locally cartesian closed, lextensive category with coequalisers and a NNO

$\implies \mathbf{Cat}(\mathcal{E})$ satisfies (1) – (4).

Conversely $\mathcal{K}$ satisfies (1) – (4) $\implies \mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ a locally cartesian closed lextensive category with coequalisers and NNO.

- ① Bourke's axioms.
- ② 2-lextensivity.
- ③ Discrete opfibrations are exponentiable.
- ④ Finite 2-colimits.

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Follows from (Street-Verity).

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$\mathcal{K}$ satisfies (1) – (2) $\rightsquigarrow$ a “small $\mathcal{F}_{\mathbf{BO}}$ -pretopos”.

Examples

For the rest of this talk, fix $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$ for $\mathcal{E}$ a locally cartesian closed, lex extensive category with coequalisers and a NNO. Examples of $\mathcal{E}$ include:

- **Set**

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- Categories of assemblies **Asm** (cf. the effective topos (Hyland))

Homotopical structure

Theorem (Everaert-Kieboom-Van der Linden)

There is a model structure on $\mathbf{Cat}(\mathcal{E})$:

- *the weak equivalences are the representable weak equivalences.*
- *the fibrations are the representable isofibrations.*
- *the cofibrations are the complemented inclusion on objects functors.*

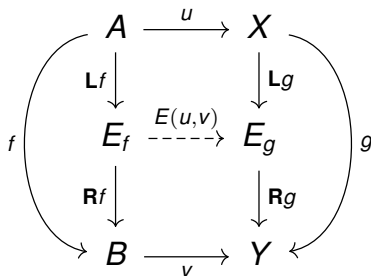
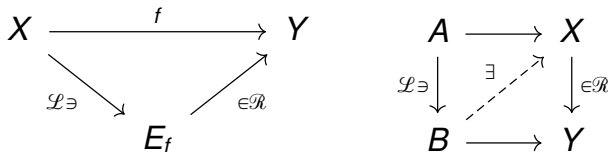
$f : \mathbb{X} \rightarrow \mathbb{Y}$ is a complemented inclusion on objects if

$$f_0 \cong \iota_{X_0} : X_0 \hookrightarrow X_0 + C.$$

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- 1 2-categorical axioms
- 2 Algebraic homotopy theory**
- 3 Type theory
- 4 Future work

Functorial wfs



giving endofunctors $\mathbf{L}, \mathbf{R} : \mathbf{C}^2 \rightarrow \mathbf{C}^2$

Relation to algebraicity

$\mathbf{L}, \mathbf{R} : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ are moreover (co)pointed!

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \mathbf{L}f \downarrow & & \downarrow f \\ \bullet & \xrightarrow{\mathbf{R}f} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\mathbf{L}f} & \bullet \\ f \downarrow & & \downarrow \mathbf{R}f \\ Y & \xlongequal{\quad} & Y \end{array}$$

$\rightsquigarrow$

$$\epsilon : \mathbf{L} \rightarrow \text{id} \qquad \text{and} \qquad \eta : \text{id} \rightarrow \mathbf{R}.$$

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An $(\mathbf{R}, \eta)$ -algebra structure for f is given by a square

$$\begin{array}{ccc}
 \bullet & \xrightarrow{\phi} & X \\
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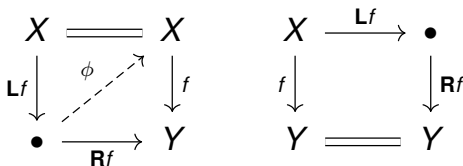
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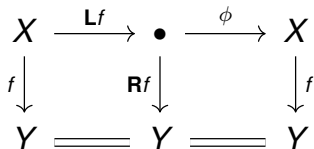
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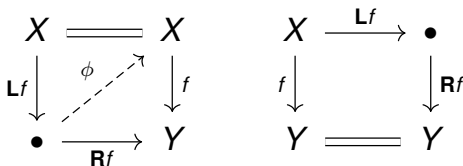
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Lemma

There exists a $(\mathbf{R}, \eta)$ -algebra structure on $f \iff f \in \mathcal{R}.$

Algebraic weak factorisation systems

Definition (Grandis and Tholen)

An *algebraic weak factorisation system* on a category $\mathbf{C}$ is a pair $(\mathbb{L}, \mathbb{R})$ of a comonad and a monad on $\mathbf{C}^2$ such that $(\overline{\mathbb{L}\text{-Coalg}}, \overline{\mathbb{R}\text{-Alg}})$ is a wfs.

Algebraic model structures

Definition (Riehl)

An *algebraic model structure* on a homotopical category $(\mathbf{C}, \mathcal{W})$ is a pair of algebraic weak factorisation systems $(\mathrm{TC}, \mathrm{F})$ and $(\mathbb{C}, \mathrm{TF})$ satisfying some conditions.

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Example

Any cofibrantly generated model structure!

Algebraic model structure

Theorem (H.)

- 1 *There is an algebraic model structure on **Cat**($\mathcal{E}$).*
- 2 *The (co)monads on this are described explicitly.*
- 3 *It has underlying model structure of (Everaert, Kieboom, Van der Linden).*
- 4 *It is cartesian monoidal.*
- 5 *It is cofibrantly generated.*
- 6 *The algebraic fibrations are the cloven isofibrations.*

Algebraic model structure

Theorem (H.)

- 1 *There is an algebraic model structure on $\mathbf{Cat}(\mathcal{E})$.*
- 2 *The (co)monads on this are described explicitly.*
- 3 *It has underlying model structure of (Everaert, Kieboom, Van der Linden).*
- 4 *It is cartesian monoidal.*
- 5 *It is cofibrantly generated.*
- 6 *The algebraic fibrations are the cloven isofibrations.*

Corollary: we can lift Everaert, Kieboom and Van der Linden's model structure to the category of $\mathbb{M}$ -modules for an internal monoidal category $\mathbb{M}$.

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The algebraic model structure on $\mathbf{Cat}(\mathcal{E})$ restricts to $\mathbf{Gpd}(\mathcal{E})$.

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The algebraic model structure on $\mathbf{Cat}(\mathcal{E})$ restricts to $\mathbf{Gpd}(\mathcal{E})$.

Such a $(2, 1)$ -category models MLTT.

Type theoretic awfs

Definition (Gambino-Larrea)

A *type-theoretic algebraic weak factorisation system* on a category $\mathbf{C}$ is an awfs $(\mathbb{T}\mathbf{C}, \mathbb{F})$ with some extra structure and satisfying certain conditions.

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Theorem (Gambino-Larrea)

Type theoretic awfs model MLTT with Σ , Π and Id -types.

The $\mathbb{F}$ -algebras model the dependent types.

Internal groupoidal model of MLTT

Theorem (H.)

The awfs $(\mathbb{T}\mathbb{C}, \mathbb{F})$ on the category $\mathbf{Gpd}(\mathcal{E})$ is equipped with the structure of a type theoretic awfs.

So cloven isofibrations form a model of MLTT.

Examples

- **Set**
- Any presheaf category $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$. Note that $\mathbf{Gpd}([\mathbb{C}^{\text{op}}, \mathbf{Set}]) \cong [\mathbb{C}^{\text{op}}, \mathbf{Gpd}]$.
- Any Grothendieck topos.
- Any elementary topos with a natural numbers object.
- Arithmetic Π -pretoposes.
- Palmgren's CETCS.
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Realisability 2-topos

In joint work with Sam Speight:

- $\mathbf{Asm}^{\text{ex/reg}} = \mathcal{E}\mathbf{ff}$.
- There is a modest discrete opfib. classifier in $\mathbf{Cat}(\mathbf{Asm})$ (cf. Weber's elementary 2-toposes).
- It is not a Grothendieck 2-topos.
- It is a 2-category with a class of small discrete opfibs.
- $\mathbf{Gpd}(\mathbf{Asm})$ models **MLTT**. In this case the classifier becomes a univalent universe of small 0-types.

Related work by Awodey-Emmenegger and Agwu and HoTTLean (Hua, Awodey, Carneiro, Hazratpour, Nawrocki, Woolfson, Xu)

Directed type theory

Together with Fernando Chu:

Cat($\mathcal{C}$) models directed type theory.

Summary

- For $\mathcal{K}$ a 2-category satisfying some axioms, $\mathcal{K}$ is of the form $\mathcal{K} \simeq \mathbf{Cat}(\mathcal{E})$ for richly structured $\mathcal{E}$.
- It therefore has an algebraic homotopy theory.
- Its $(2, 1)$ -core models higher dimensional logic i.e MLTT.
- Such a thing should be an example of a small $(2, 1)$ -pretopos.

Arxiv: The algebraic internal groupoid model of Martin-Löf type theory, 2025.



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