Towards a Foundation for General Systems Theory

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Towards a double operadic theory of systems
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Strict Discrete Opfibration Classifier for 2-algebraic structures
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Matteo Capucci

Loose Bimodules and their construction
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Double functorial representation of Indexed Monoidal Structures
José Siqueira

General Systems Theory

- A system is a complex structure formed by interacting component systems

- Tasks of General Systems Theory:
  1. The design of complex systems
  2. The analysis of system behavior
  3. The specification of desirable system behavior and the analysis of its satisfaction

... all with a focus on compositionality.

Compositionality

- A system is a complex structure formed by interacting component systems
- Focus on the compositionality of our tasks

① Modular design through interacting components

② Analysis of composite behavior in terms of component behavior

③ Assume-Guarantee reasoning about the satisfaction of component specs, assuming that the guarantees provided by other interacting components are met.

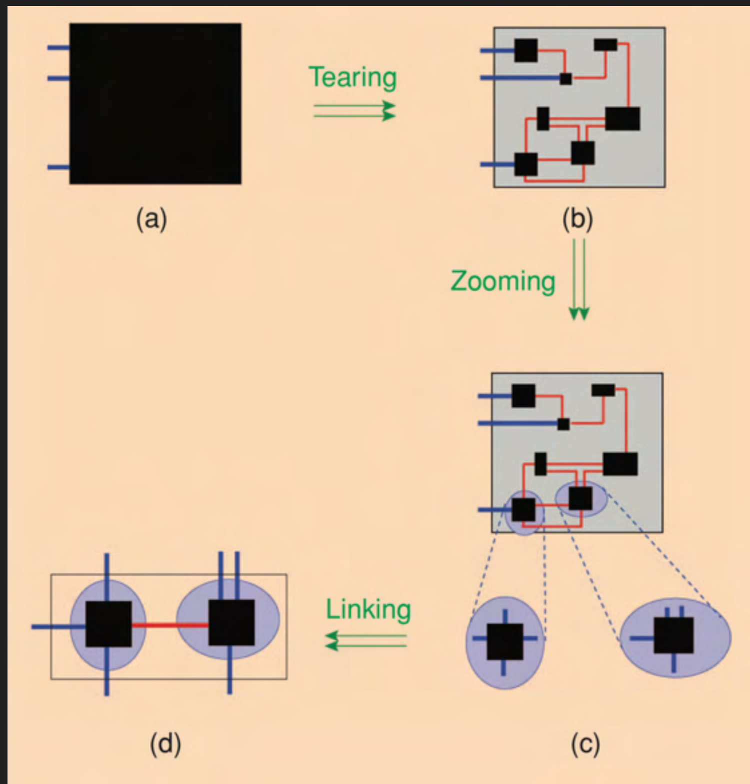
Examples of Systems Theories

- Systems of Ordinary Differential Eqns
- Machines & Automata
- Markov decision processes
- System "diagrams"
  - Circuit diagrams
  - Stock-Flow diagrams
  - Flow charts / Transition Systems
- Petri Nets
- Hamiltonians / Lagrangians
- Partial Differential Eqns

⋮

Jan Willems' Behavioral Control Theory

- Systems are defined as time-varying sets of behaviors.
- Systems compose by sharing variables
- "Zooming, Tearing, Linking"...



Categorical Systems Theories

- A long history, but...
- One key example:

$$\begin{tikzcd} TS \arrow[d] & r^*I \arrow[l, dashed, "u"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d] \\ S \arrow[r, equal] & S \arrow[r, dashed, "r"] & O \end{tikzcd}$$

DYNAMICAL SYSTEMS AND SHEAVES

PATRICK SCHULTZ, DAVID I. SPIVAK, AND CHRISTINA VASILAKOPOULOU

① Systems of ODEs interact by coupling

ⓐ there is an operad (symmetric multicat) $\mathcal{W}_c$ of ODE interfaces and coupling laws

ⓑ Systems of ODEs form an algebra $\mathbf{ODE} : \mathcal{W}_c \to \mathbf{Cat}$

② Solutions of ODEs are sheaves on a site $\mathbb{IR}$ of time intervals

ⓐ Solving ODEs gives a morphism of operad algebras: $\mathcal{W}_c \longrightarrow \mathcal{W}_D$

i.e. System S w/ interface I maps to

$\{\text{Behaviors of } S\} \longrightarrow \{\text{Behaviors of } I\}$

$$\begin{tikzcd} \mathcal{W}_c \arrow[rr] \arrow[dr, "\mathbf{ODE}"'] & \Rightarrow & \mathcal{W}_D \arrow[dl, "\mathbf{Span}(\mathbf{Sh}(\mathbb{IR}))"] \\ & \mathbf{Cat} & \end{tikzcd}$$

③ Specifications are predicates of the sheaves of behaviors

ⓐ The internal logic of $\mathbf{Sh}(\mathbb{IR})$ has temporal modalities

Double Operadic Systems Theory

Takeaways:

any kind of system!

① Systems and their maps organize into algebras for

CAT-operads (Weber)

double operads of interactions.

② Behaviors of systems are often representable giving (lax) morphisms

Willems' "behavioral" Systems theory

$$\mathrm{Sys} \xrightarrow{\mathrm{Sys}(c,+)} \mathrm{Span}(\mathrm{Set})$$

③ Families of representables indexed by categories of "clocks" $\mathcal{C}$ land in sheaves on clocks.

$$\mathrm{Sys} \xrightarrow{\mathrm{Sys}(c-,+)} \mathrm{Span}(\mathrm{Sh}(\mathcal{C},j))$$

The internal logic of these sheaves support modalities, giving a "temporal" logic for specifications of these systems.

Analogy with Mathematical Logic

Category Theory	Model Theory	(General) Systems Theory
structure-preserving map	element	behavior
structured object	model	system
structured category	theory	theory of systems
2-monadic 2-category	doctrine	doctrine

Logical Doctrines

- A doctrine is a 2-functor

$$\text{Theory} \xrightarrow{\text{Mod}} \text{Model}$$

Both 2-algebraic (Alg(T) for a 2-monad T), but

- Theory is of KZ-type
- Model might not be

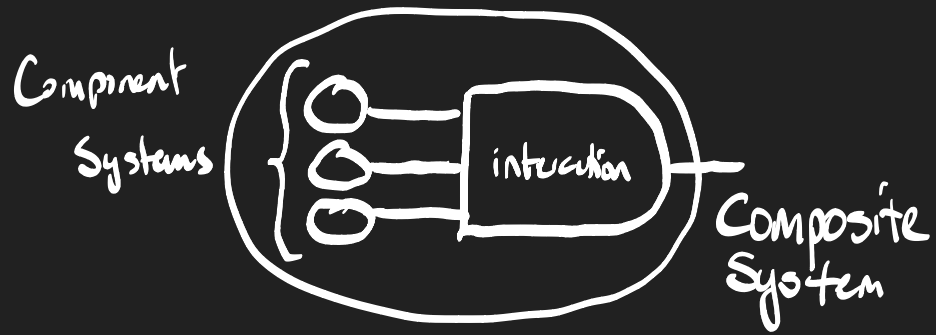
E.g.

$$\text{Coherent} \longrightarrow \text{Ultracat}$$

$$\text{Lex} \longrightarrow \text{RFP}$$

Doctrines of Systems Theories

- Composition of systems is via <u>interactions</u>
  - interactions must be specified as part of the systems theory



$$
\begin{tikzcd}
\text{Systems Theory} \arrow[r, "\mathbb{S}"] \arrow[d] & \text{2-Algebra of Systems, interacting} \arrow[d] \\
\text{Interaction Theory} \arrow[r, "\mathbb{I}"'] & \text{2-Algebra for Interactions}
\end{tikzcd}
$$

Why is 2-algebra important?

- Suppose $\mathcal{K}$ is a 2-cat w/ pullbacks of discrete opfibrations and a discrete opfibration classifier $\Omega \in \mathcal{K}$

Thm (Capucci - M.): For any 2-monad $T : \mathcal{K} \to \mathcal{K}$, cartesian along and preserving discrete opfibrations, then the classifying map

$$\begin{tikzcd} T\Omega_\bullet \arrow[r, "\omega_\bullet"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, "u"] \\ T\Omega \arrow[r, "\omega"'] & \Omega \end{tikzcd}$$

becomes a strict* T-hom

and it classifies strict discrete opfibrations in $\mathrm{Alg}^{lax}(T)$.

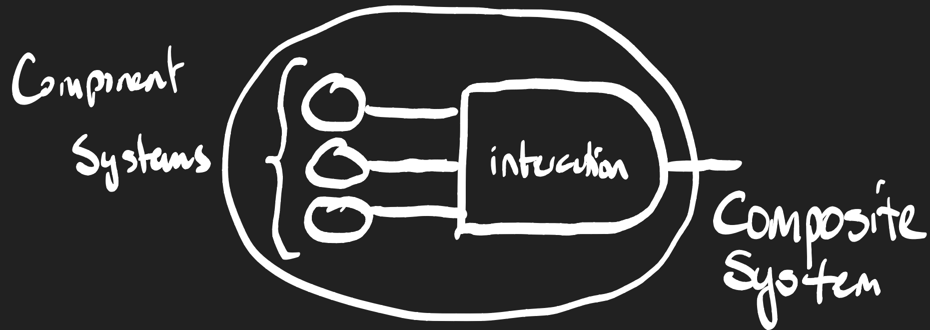
- If $\mathcal{K} = [\mathcal{G}^{op}, \mathrm{Cat}]$, then $\Omega(g) = [(\mathcal{G} \downarrow g)^{op}, \mathrm{Set}]$ is an algebra of generalized spans

Mesiti

$\Rightarrow$ representable features of systems compose as generalized spans!

2-algebra of Systems Theories

Sorts:



and maps between them.

E.g.

① Algebras for Double Operads (CAT-operads)

② Symmetric Monoidal loose right modules of symmetric monoidal double categories.

Conjecture: The 2-categories of isofibrant ② and isofibrant + representable ① and pseudo-morphisms are equivalent

! but not (co)lax !

Symmetric Monoidal Loose Right Modules

- Double categories are pseudo-categories in $\mathbf{Cat}$

$$\mathbb{D}_0 \xleftarrow{s} \mathbb{D}_1 \xrightarrow{t} \mathbb{D}_0$$

Symmetric Monoidal Loose Right Modules

- Double categories are pseudo-categories in Cat

$$
\begin{tikzcd}
 & & \mathbb{D}_0 & & \mathbb{D}_0 \arrow[ddd, equal] \\
M \arrow[urr] & & & \mathbb{D}_1 \arrow[ul, "s"'] \arrow[ur, "t"] & \\
 & M \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[ul] \arrow[urr] \arrow[d, "\alpha"'] & & & \\
 & M \arrow[rrr] & & & \mathbb{D}_0
\end{tikzcd}
$$

Loose right modules are right pseudo-modules

Symmetric Monoidal Loose Right Modules

- Double categories are pseudo-categories in **Cat**
- Loose right modules are right pseudo-modules

Quick "double barrel" definition:

$$\{0 \nrightarrow 1\}$$

"walking loose arrow"

$$\begin{tikzcd}
\ell\mathrm{Mod}_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}\mathrm{bl} \downarrow \mathrm{Loose} \arrow[d, "0^*"] \\
* \arrow[r, "*"'] & \mathbb{D}\mathrm{bl}
\end{tikzcd}$$

$\ell\mathrm{Mod}_r$ is a cartesian 2-category.

Thm (Brown-Carlson-Libkind-M.):

These two definitions of loose right module agree

Proof: Via pseudo-models of $\mathcal{F}$-sketches

↑ new!

Symmetric Monoidal Loose Right Modules

- Double categories are pseudo-categories in Cat
- Loose right modules are right pseudo-modules

Quick "double barrel" definition:

$$
\begin{tikzcd}
\ell\mathrm{Mod}_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Dbl} \downarrow \mathrm{Loose} \arrow[d, "0^*"] \\
* \arrow[r, "*"'] & \mathrm{Dbl}
\end{tikzcd}
$$

$\{0 \nrightarrow 1\}$ "walking loose arrow"

$\ell\mathrm{Mod}_r$ is a cartesian 2-category.

Def: A symmetric monoidal loose right module is a symmetric pseudo-monoid in $\ell\mathrm{Mod}_r$

Constructing Loose Right Modules

- $\mathbb{L}\mathrm{oose} \times - : \mathrm{Dbl}^{\times} \longrightarrow \mathrm{Dbl} \downarrow \mathbb{L}\mathrm{oose}$ gives loose hom bimodules.

Thm (Brown – Carlson – Libkind – M.): Restriction of loose bimodules gives a cartesian pseudo-functor

$$\mathrm{Niche} \xrightarrow{\times} \mathrm{Dbl} \downarrow \mathbb{L}\mathrm{oose}$$

Cor: Restriction of symmetric monoidal loose right modules along lax symmetric monoidal double functors whose unitors and laxitors are commuter transformations is still symmetric monoidal.

E.g.

$$\begin{tikzcd}
* \arrow[r, "\mathbb{D}(\mathbb{1}{,}\, i-)", "|"{marking}, dashed] \arrow[d, "\mathbb{1}"'] \arrow[dr, phantom, "\mathrm{res.}"] & \mathbb{I} \arrow[d, "i"] \\
\mathbb{D} \arrow[r, "\mathbb{D}(-{,}-)"', "|"{marking}] & \mathbb{D}
\end{tikzcd}$$

Examples of System Doctrines (Libkind - M.)

① "parameter setting" generalized Moore machines

E.g.
Moore machines
ODEs
Markov decision Processes
"Open" Coalgebras

"notion of tangent bundle" $T \left(\begin{array}{c} \mathcal{E} \\ \downarrow \pi \\ B \end{array} \right) \longmapsto \mathbf{Lens}_{\mathcal{E}}(T-,-)$

Spivak-lenses

Span($\mathcal{E}$, (vert, cart))

$$\begin{tikzcd} \mathbf{Tangency} \arrow[r] \arrow[d] & \mathbf{lMod}_r \arrow[d] \\ \mathbf{Fib} \arrow[r, "\mathbf{Lens}"'] & \mathbf{Dbl} \end{tikzcd}$$

Note: Every cartesian category has a simple fibration

Lenses in the simple fibration of a free cartesian cat are directed wiring diagrams

So we recover the Schultz-Spivak-Vasilakopoulou algebras

Examples of System Doctrines (Libkind – M.)

② "Port-plugging" Structured or Decorated Cospans (Fong, Baez, Courser, Patterson...)

E.g.
- Petri Nets
- Labeled transition Systems
- Stock-Flow diagrams
- Causal Loop diagrams
- Circuit diagrams

$$G \xrightarrow{i} UC \longmapsto \mathbf{Cospan}(C)(\emptyset, i-)$$

$$\begin{tikzcd} \mathsf{Cat} \downarrow U_{Rex} \arrow[r] \arrow[d] & l\mathsf{Mod}_r \arrow[d] \\ \mathsf{Cat} \arrow[r, "\mathrm{Cospan}(F_{Rex})"'] & \mathbb{D}\mathsf{bl} \end{tikzcd}$$

a "hypergraph double cat"

Note: Cospans in free Rex categories are "undirected wiring diagrams"

Thm (Fong–Spivak): Algebras for such cospan operads are **hypergraph categories**

Specifications & Graphical Regular Logic (Clingman–Fong–Spivak)

Specifications compose by sharing variables, like the behaviors of systems

Thm (Siqueira): A monoidal regular hyperdoctrine $\mathcal{C}tx \xrightarrow{\text{Pred}} \text{Pos}$ corresponds to a lax monoidal double functor $\text{Span}(\mathcal{C}tx)^{op} \xrightarrow{\text{Pred}'} \mathbb{Q}t(\text{Pos})$ whose unitors + laxitors are companion commuters (equiv. to Frobenius)

If $\mathcal{C}tx = (\text{Fin} \downarrow T)^{op}$ is free lex on a set of "types" T, then

$$\begin{tikzcd}
\ast \arrow[rr, "\mathbb{Q}t(\text{Pos})(\ast\text{, } \text{Pred}'(-))", "|" marking] \arrow[d] & & \text{Cospan}(\text{Fin}\downarrow T) = \text{Span}((\text{Fin}\downarrow T)^{op})^{op} \arrow[d, "\text{Pred}'"] \\
\mathbb{Q}t(\text{Pos}) \arrow[rr, "\mathbb{Q}t(\text{Pos})(-,-)"', "|" marking] \arrow[urr, phantom, "\text{res}"] & & \mathbb{Q}t(\text{Pos})
\end{tikzcd}$$

"graphical regular logic"

Děkuji