Bicolimit Presentations of Type Theories

Vít Jelínek
(jww John Bourke)

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UNIVERSITY OF SUSSEX

Overview

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We take inspiration from universal algebra – to *present* an algebra, we can give its *presentation* (generators + relations). Categorically – colimit of free algebras.

Goal of this talk:
Show that we can construct examples of type theories via bicolimits of free type theories + explain how this interacts with semantics

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What needs to be captured?

Types (living in a context)

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Terms (living in a context and a type)

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Types (living in a context)

Terms (living in a context and a type)

Contexts (can be extended)

Natural Models of DTT

Definition (Representable Natural Transformation) [Algebraic geometers]

Let F, G be presheaves over a category $\mathcal{C}$. Then a natural transformation $\alpha \colon F \to G$ is called *representable* if, for every $\beta \colon \text{よ} c \to G$, the pullback

$$\begin{array}{ccc} \bullet & \longrightarrow & F \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \alpha} \\ \text{よ} c & \xrightarrow{\beta} & G \end{array}$$

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Definition (Natural Model) [Awodey/Fiore]

A *natural model* in a category $\mathcal{C}$ with a terminal object is a representable natural transformation $p\colon Tm \to Ty$.

The same as CwA, CwF.

Explanation of Natural Models

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the object representing the pullback of $A \colon \text{よ}\Gamma \to Ty$ along p is seen as the context extension $\Gamma.A$

Unit Types

A type theory has unit types if we have symbols $\mathbb{1}, \star$ together with the following rules:

$$\frac{}{\Gamma \vdash \mathbb{1}\ \mathit{Ty}}\ \mathbb{1}\text{-form} \qquad \frac{}{\Gamma \vdash \star : \mathbb{1}}\ \mathbb{1}\text{-intro}$$

$$\frac{\Gamma \vdash t : \mathbb{1}}{\Gamma \vdash t \equiv \star : \mathbb{1}.}\ \mathbb{1}\text{-}\eta$$

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Definition (Natural Models with Unit Types) [Folklore?]

A *natural model with unit types* is a natural model $p \colon \mathit{Tm} \to \mathit{Ty} \in \mathbf{Set}^{\mathcal{C}^{op}}$ together with maps $1 \xrightarrow{\mathbb{1}} \mathit{Ty}, 1 \xrightarrow{\star} \mathit{Tm}$ such that the following square is a pullback:

$$\begin{array}{ccc} 1 & \xrightarrow{\star} & \mathit{Tm} \\ {\scriptstyle id}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{\mathbb{1}} & \mathit{Ty}. \end{array}$$

Other Versions of DTT?

What about other constructors? $(0, \Pi, \Sigma, \mathbb{N}, \ldots)$

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Can we have a parametric definition of semantics?

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Can we have a parametric definition of semantics?

First we need a definition of type theory!

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Categories with Representable Maps

Definition (Category with Representable maps) [Uemura]

A *category with representable maps (CwR)* is a category $\mathcal{C}$ with finite limits and a class of *representable* maps $R \subseteq \mathcal{C}^{\to}$ that

- is closed under compositions and contains every isomorphism;
- is pullback-stable;
- are exponentiable.

Maps in R will be denoted by $\rightarrowtriangle$.

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Main example: $\mathbf{Set}^{\mathcal{C}^{op}}$ with representable natural transformations.

CwRs Explanation

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Let $\mathcal{C}$ be a CwR, then...

- its objects represent judgement forms (Ty, $Tm, \ldots$);
- arrows are derivations;
- limits are used to create more complicated judgements ($\Gamma \vdash J_1 \; \Gamma \vdash J_2$, empty judgement, $\ldots$);
- representable arrows are used to describe judgements that can appear in contexts and exponentials along those are used to bind variables (moving the judgements in contexts).

Functorial Semantics

Definition (Model of a CwR) [Uemura]

A *model of a CwR* $\mathcal{T}$ consists of a category $\mathcal{C}$ with a terminal object and a CwR functor $M\colon \mathcal{T} \to \mathbf{Set}^{\mathcal{C}^{op}}$.

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- Let $NM_{\mathbb{1},\star,\eta}$ be the CwR freely generated by

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  Then its models are natural models with unit types.

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What does it mean to freely generate? Is it always possible?

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2-Category of Type Theories

Definition (Rep) [Uemura]

We denote **Rep** the 2-category that has

- 0-cells. . . small CwRs;
- 1-cells. . . functors preserving all the CwR structure;
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Type theories can be glued!

Generators

Definition (Marked Category with Squares) [Bourke & J.]

A *marked category with squares* is a category $\mathcal{C}$ equipped with a class of arrows $M \subseteq \mathcal{C}^{\rightarrow}$ and a class of commutative squares $S \subseteq Sq(\mathcal{C})$ such that any square whose domain and codomain are isos is in S, and both arrows and squares are closed under composition.

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Definition ($\mathbf{Cat}_m$) [Bourke & J.]

We denote $\mathbf{Cat}_m$ the 2-category that has

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Theorem ($\mathbf{Cat}_m$ is nice) [Bourke & J.]

$\mathbf{Cat}_m$ is an accessible 2-category with all 2-limits and 2-colimits.

Free Generation

We have a forgetful 2-functor $U\colon \mathbf{Rep} \to \mathbf{Cat}_m$ sending $\mathcal{T}$ to $\mathcal{T}$ with representable maps and pullback squares.

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Theorem (U is nice) [Bourke & J.]

U preserves directed colimits and flexible limits.

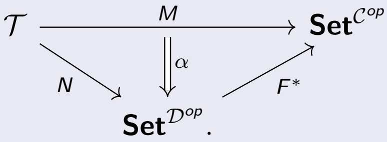
Corollary (U is nicer) [Bourke, Lack, Vokřínek]

U has a left biadjoint $F\colon \mathbf{Cat}_m \to \mathbf{Rep}$.

Categories of Models

Definition (Category of Models of a CwR)

Let $\mathcal{T}$ be a CwR, its *category of models* $Mod(\mathcal{T})$ has models of $\mathcal{T}$ as objects and a morphism from $M\colon \mathcal{T} \to \mathbf{Set}^{\mathcal{C}^{op}}$ to $N\colon \mathcal{T} \to \mathbf{Set}^{\mathcal{D}^{op}}$ is a terminal object preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ together with a natural transformation α



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$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\quad M \quad} & \mathbf{Set}^{\mathcal{C}^{op}} \\ & {\scriptstyle N} \searrow \quad \Downarrow \alpha \quad \nearrow {\scriptstyle F^*} & \\ & \mathbf{Set}^{\mathcal{D}^{op}}. & \end{array}$$

satisfying some form of the Beck-Chevalley condition.

We have a functor $Mod(\mathcal{T}) \to \mathbf{CatT}$ ($\mathbf{CatT}$ are categories with a terminal object) sending $(M, \mathcal{C})$ to $\mathcal{C}$ and (α, F) to F.

Interaction of Free Constructions and Semantics

We have a 2-functor $Mod \colon \mathbf{Rep}^{op} \to \mathbf{CAT}/\mathbf{CatT}$ that sends a type theory to its category of models.

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We have also models of marked categories with squares:

Definition (Model of a Marked Category with Squares)

A *model of* $\mathcal{C} \in \mathbf{Cat}_m$ in a category $\mathcal{D}$ with a terminal object is a $\mathbf{CAT}_m$ functor $\mathcal{C} \to \mathbf{Set}^{\mathcal{D}^{op}}$ (where marked arrows are the representable natural transformations and squares are pullback squares).

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Theorem [Bourke & J.]

For every $\mathcal{C} \in \mathbf{Cat}_m$, we have $Mod(F\mathcal{C}) \simeq Mod(\mathcal{C})$.

Examples of Type Theories I

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- Set $NM := F(Tm \overset{p}{\rightarrow} Ty)$, then $Mod(NM)$ is equivalent to natural models.
- Models of the following bipushout in **Rep**

$$
\begin{array}{ccc}
F(a \quad b \xrightarrow{\alpha} c) & \xrightarrow{(a \mapsto 1,\, \alpha \mapsto p)} & NM \\
\downarrow & & \downarrow \\
F(a \to b \to c) & \xrightarrow{\qquad\qquad} & NM_{\mathbb{1},\star}
\end{array}
$$

are natural models with two maps $\star \colon 1 \to Tm$ and $\mathbb{1} \colon 1 \to Ty$ such that $p\star = \mathbb{1}$.

Examples of Type Theories II

- Let $\mathcal{C} \in \mathbf{Cat}_m$ be the free commutative square and $\mathcal{D}$ the free marked commutative square. Then models of the following bipushout in **Rep**

$$\begin{array}{ccc} F\mathcal{C} & \xrightarrow{f} & NM_{\mathbb{1},\star} \\ \downarrow & & \downarrow \\ F\mathcal{D} & \longrightarrow & \ulcorner NM_{\mathbb{1},\star,\eta}, \end{array}$$

where f is the map choosing the square $\begin{array}{ccc} 1 & \xrightarrow{\star} & Tm \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{\mathbb{1}} & Ty \end{array}$, are natural

models with unit types.

Work in Progress/Wishes

- The forgetful 2-functor $U\colon \mathbf{Rep} \to \mathbf{Cat}_m$ is pseudomonadic and the induced pseudomonad is colax-idempotent.
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Thank you for your attention!