Representables in fuzzy category theory

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GL-monoid

Definition

A GL-monoid is a complete lattice enriched a further binary operation $*$, i.e., a triple $(L, \leq, *)$ such that:

1. $*$ is monotone, i.e., $\alpha \leq \beta$ implies $\alpha * \gamma \leq \beta * \gamma \ \forall \alpha, \beta, \gamma \in L$;
2. $*$ is commutative, i.e., $\alpha * \beta = \beta * \alpha \ \forall \alpha, \beta \in L$;
3. $*$ is associative, i.e., $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma \ \forall \alpha, \beta, \gamma \in L$;
4. $(L, \leq, *)$ is integral, i.e., 1_L acts as the unity: $\alpha * 1_L = \alpha \ \forall \alpha \in L$;
5. 0_L acts as the zero element in $(L, \leq, *)$, i.e. $\alpha * 0_L = 0_L \ \forall \alpha \in L$;
6. $*$ is distribute over arbitrary joins, i.e., $a * \left(\bigvee_j \beta_j\right) = \bigvee_j (a * \beta_j) \ \forall \alpha \in L, \forall \{\beta_j \mid j \in J\} \subseteq L$;
7. $(L, \leq, *)$ is divisible, i.e., $\alpha \leq \beta$ implies the existence of $\gamma \in L$ such that $\alpha = \beta * \gamma$.

L-fuzzy category

Definition

An *L-fuzzy category* is a quintuple $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ where $(\mathcal{O}b(\mathcal{C}), \mathcal{M}(\mathcal{C}), \circ)$ is a classical category and $\omega\colon \mathcal{O}b(\mathcal{C}) \to L$, $\mu\colon \mathcal{M}(\mathcal{C}) \to L$. Additionally ω and μ satisfy the following conditions:

1. if $f\colon X \to Y$, then $\mu(f) \leq \omega(X) \wedge \omega(Y)$;
2. $\mu(g \circ f) \geq \mu(g) * \mu(f)$ whenever $g \circ f$ is defined;
3. if $\mathrm{id}_X\colon X \to X$ is the identity morphism, then $\mu(\mathrm{id}_X) = \omega(X)$.

Crisp categories from fuzzy categories

Let $\mathcal{O}b_\alpha(\mathcal{C}) = \{X \in \mathcal{O}b(\mathcal{C}) \mid \omega(X) \geq \alpha\}$ be the set of α-objects and $\mathcal{M}_\alpha(\mathcal{C}) = \{f \in \mathcal{M}(\mathcal{C}) \mid \mu(f) \geq \alpha\}$ the set of α-morphisms.

Given an L-fuzzy category $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ one can obtain a crisp category, called a threshold category, by taking any idempotent element ι of L and setting $C_\iota = (\mathcal{O}b_\iota(\mathcal{C}), \mathcal{M}_\iota(\mathcal{C}), \circ)$.

In the case when $\iota = \bot = 0_L$ we get the crisp category $C_\bot = (\mathcal{O}b(\mathcal{C}), \mathcal{M}(\mathcal{C}), \circ)$ which is called the (*crisp*) bottom frame of $\mathcal{C}$.

In the case when $\iota = \top = 1_L$ we get the crisp category $C_\top = (\mathcal{O}b(\mathcal{C}), \mathcal{M}(\mathcal{C}), \circ)$ which is called the (*crisp*) top frame of $\mathcal{C}$.

Fuzzy category $\boldsymbol{FL} - \mathbf{Set}$

Objects of the fuzzy category $\boldsymbol{FL} - \mathbf{Set}$ are L-sets, i. e. pairs $\mathcal{X} = (X, A)$, where X is a crisp set and $A\colon X \to L$ is its fuzzy subset. There is a morphism $f \in \mathcal{M}(\mathcal{X}, \mathcal{Y})$ between $\mathcal{X}$ and $\mathcal{Y}$, where $\mathcal{Y} = (Y, B)$, if and only if the map $f\colon X \to Y$ satisfies $B \circ f \geq A$. The membership degrees are defined as:

$$\omega(\mathcal{X}) = \bigwedge_{x \in X} A(x),$$

$$\mu(f) = \omega(\mathcal{X}) \wedge \omega(\mathcal{Y}).$$

The top frame of this category is **Set**, but the bottom frame is$\boldsymbol{L} - \mathbf{Set}$.

Functors

Definition

Let $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega_{\mathcal{C}}, \mathcal{M}(\mathcal{C}), \mu_{\mathcal{C}}, \circ)$ and $\mathcal{D} = (\mathcal{O}b(\mathcal{D}), \omega_{\mathcal{D}}, \mathcal{M}(\mathcal{D}), \mu_{\mathcal{D}}, \circ)$ be L-fuzzy categories and let $F_1 \colon \mathcal{O}b(\mathcal{C}) \to \mathcal{O}b(\mathcal{D})$ and $F_2 \colon \mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{D})$ be maps. The quadruple $F = (\mathcal{C}, \mathcal{D}, F_1, F_2)$ is a called a δ-functor from $\mathcal{C}$ to $\mathcal{D}$ ($F \colon \mathcal{C} \to \mathcal{D}$) provided the following properties are satisfied:

1. $f \in \mathcal{M}_{\mathcal{C}}(X, Y)$ implies $F_2(f) \in \mathcal{M}_{\mathcal{D}}\left(F_1(X), F_2(X)\right)$;

2. F preserves identities, i.e, $F_2(\mathrm{id}_X) = \mathrm{id}_{F_1(X)}$ for any $X \in \mathcal{O}b(\mathcal{C})$;

3. F_2 preserves composition, i.e. $F_2(g \circ f) = F_2(g) \circ F_2(f)$ provided the composition $g \circ f$ is defined;

4. $\mu_{\mathcal{C}}(f) * \delta \leq \mu_{\mathcal{D}}(F_2(f))$ for any $f \in \mathcal{M}(\mathcal{C})$.

L-set category

Theorem [A. Pultr, 1976]

Let $Q = (L, \leq, *, 1_L)$ be a quantale, where * is a t-norm, then the structure $(\boldsymbol{L} - \mathbf{Set}, \otimes, (\{\star\}, 1))$ is a symmetric monoidal closed category, where $\otimes$ is Cartesian product, that is matched with the t-norm *.

Representable $\mathcal{V}$-enriched functor

Let $\mathcal{C}$ be a $\mathcal{V}$-category, where $\mathcal{V}$ is a symmetrical monoidal closed category. A $\mathcal{V}$-enriched functor $F\colon \mathcal{C} \to \mathcal{V}$ is called *representable* if there is a $K \in \mathcal{O}b(\mathcal{C})$ and a $\mathcal{V}$-enriched natural transformation $\eta\colon F \Longrightarrow \mathcal{C}(K, -)$.

Functors in threshold categories

Proposition

If $F\colon \mathcal{C} \to \mathcal{D}$ is a δ-functor and δ is an idempotent element, then the restriction of F to the threshold categories $F_\delta\colon \mathcal{C}_\delta \to \mathcal{D}_\delta$ is a crisp functor between the corresponding categories.

Proposition

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a 0-functor if and only if the restriction of F to the bottom frame categories $F_\perp\colon \mathcal{C}_\perp \to \mathcal{D}_\perp$ is a crisp functor between the corresponding categories.

Fuzzy functor representation

Suppose $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ is a fuzzy category and $F\colon \mathcal{C} \to \boldsymbol{FL}\textbf{-Set}$ is an α-functor and there is an α-object $K \in \mathcal{O}b(\mathcal{C})$ and an α-natural transformation $\eta\colon F \Longrightarrow \mathrm{Hom}_{\mathcal{C}}(K, -)$.

Then for each idempotent $\iota \leq \alpha$

1. $F_\iota\colon \mathcal{C}_\iota \to \boldsymbol{FL}\textbf{-Set}_\iota$ is a functor;
2. $K \in \mathcal{O}b(\mathcal{C}_\iota)$;
3. $\eta_\iota\colon F_\iota \Longrightarrow \mathrm{Hom}_{\mathcal{C}_\iota}(K, -)$ is a natural transformation

Fuzzy functor representation

Since $\mathcal{C}$ is a fuzzy category, the threshold category $\mathcal{C}_\iota$ can inherit morphism membership degrees from $\mathcal{C}$.

Each morphism from $\mathrm{Hom}_{\mathcal{C}_\iota}(A, B)$ can be assigned a value from the lattice L.

This means that if $\mathcal{C}$ is a locally small category, then $\mathrm{Hom}_{\mathcal{C}_\iota}(A, B)$ can be considered as a hom-object from the monoidal category $\boldsymbol{L} - \mathbf{Set}$.

Fuzzy functor representation

Theorem

Suppose $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \omega, \mathcal{M}(\mathcal{C}), \mu, \circ)$ is a fuzzy category and $F\colon \mathcal{C} \to \boldsymbol{FL}\text{-}\mathbf{Set}$ is an α-functor and there is an α-object $K \in \mathcal{O}b(\mathcal{C})$ and an α-natural transformation $\eta\colon F \Longrightarrow \mathrm{Hom}_{\mathcal{C}}(K, -)$. Then for each idempotent $\iota \leq \alpha$ the functor F_ι is a representable $\boldsymbol{L} - \mathbf{Set}$-functor when $\mathbb{C}_\iota$ is viewed as an enriched category over $\boldsymbol{L} - \mathbf{Set}$.

References

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Thank you for your attention!