Positively closed topos-valued models

Kristóf Kanalas

2025

Motivation

Set-valued models (of coherent theories):

groups, rings, local rings, fields, difference fields, modules, lattices, graphs, sets with an equivalence relation,. . .

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$\mathcal{E}$-valued models (of coherent theories):

T	$\mathcal{E}$	T-models in $\mathcal{E}$
Abelian groups	$Sh(X)$	sheaf of Abelian groups
local rings	$Sh(X)$	locally ringed spaces
rings/ fields	$\mathbf{Set}^{\circlearrowleft}$	difference rings/ fields
arbitrary	$Sh(B, \tau_{can})$	models in V^B

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Plan

Develop topos-valued positive model theory.

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Develop topos-valued positive model theory.

Tool

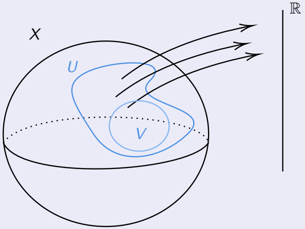
External approach to internal models. (Calculus of coherent functors.)

Internal models in $Sh(X)$.

A sheaf of structures

A sheaf of structures

Example



$U \mapsto Cont(U, \mathbb{R})$

$V \subseteq U \mapsto Cont(U, \mathbb{R}) \xrightarrow{restr} Cont(V, \mathbb{R})$

$\rightsquigarrow Open(X)^{op} \to \mathbf{Set}$

moreover:

Given $U = \bigcup V_i$ and $f_i : V_i \to \mathbb{R}$ compatible, they glue together to a unique $f : U \to \mathbb{R}$.

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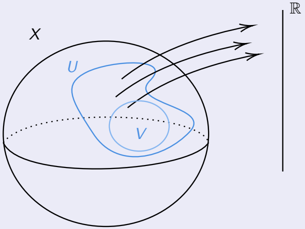

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Definition

L signature. A sheaf of L-structures on X is a functor

$$M : Open(X)^{op} \to \mathbf{L-str}$$

s.t. the same gluing condition holds.

Example: $Cont(-, \mathbb{R})$ is a sheaf of L_{Ring}-structures, with pointwise $0, 1, -, +, \cdot$

A sheaf of models

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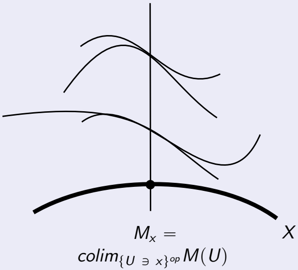
Stalk at x.



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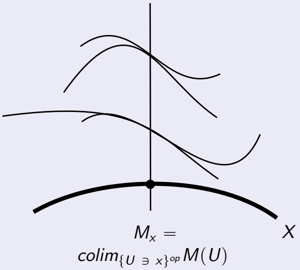

Definition

$T \subseteq L^{g}_{\omega\omega}$. A sheaf of L-structures $M : Open(X)^{op} \to \mathbf{L} - \mathbf{str}$ is a model of T if for each $x \in X$ we have $M_x \models T$.

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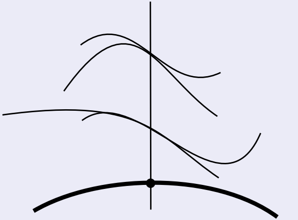

Example

T is the theory of comm. local rings:
comm. ring & $0 = 1 \Rightarrow \bot$ &
$\top \Rightarrow \exists y : x \cdot y = 1 \vee \exists y : (1 - x) \cdot y = 1$

T-model in $Sh(X)$: locally ringed space.

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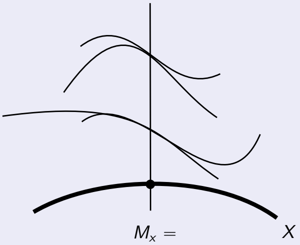

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(E.g. study positively closed locally ringed spaces, etc.)

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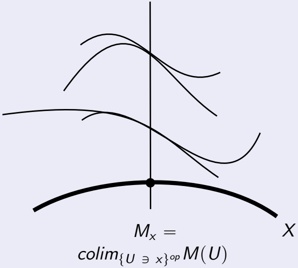

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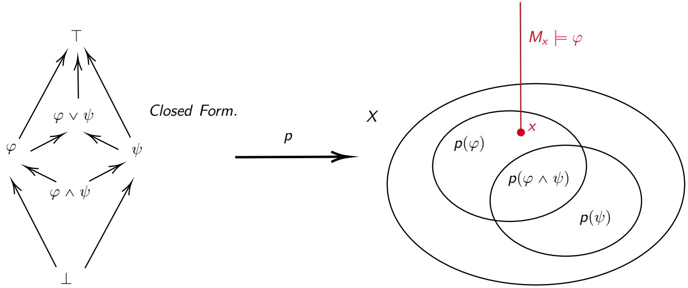
External approach to internal models:

T-models in $Sh(X) =$
coherent functors $\mathcal{C}_T \to Sh(X)$

Translating between the internal and the external view.

Can we prescribe which statements should be true in which fibers?



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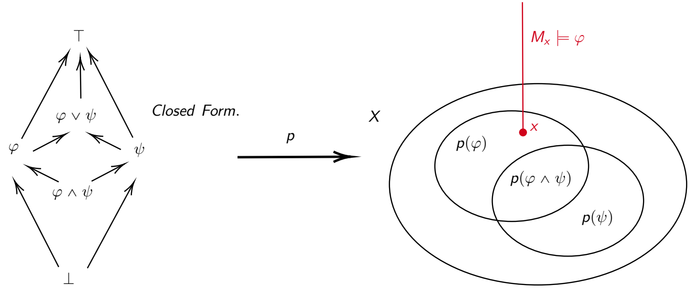
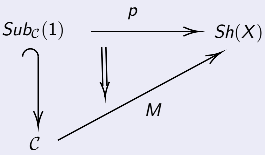


Definition: $Sh(X)$ can realize types, if

for any small coherent cat. $\mathcal{C}$ and coherent functor $p : Sub_{\mathcal{C}}(1) \to Sh(X)$, there is a coherent lift:

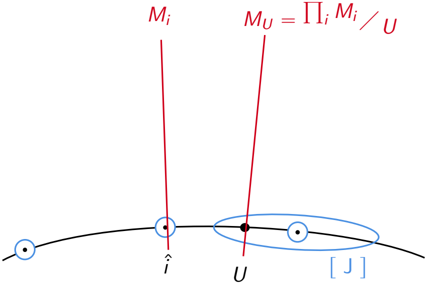


Theorem

If X is an extremally disconnected Stone-space (i.e. $X = Spec(B)$ for a complete Boolean algebra) then $Sh(X)$ can realize types.

Proof idea: we can assume $B = 2^I$.

At principal ultrafilters we find arbitrary solutions, at non-principal ones we take the ultraproduct.



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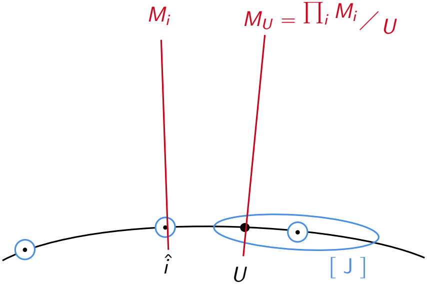

Question

Can every Grothendieck topos realize types?

Positively closed models.

A lattice invariant of $Sh(X)$-valued models

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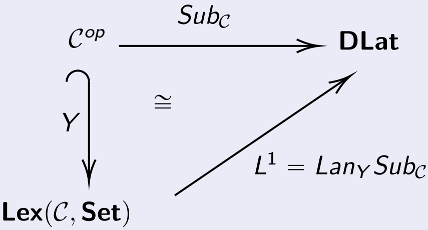

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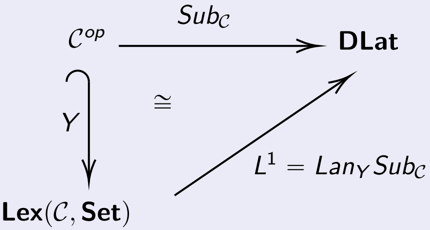

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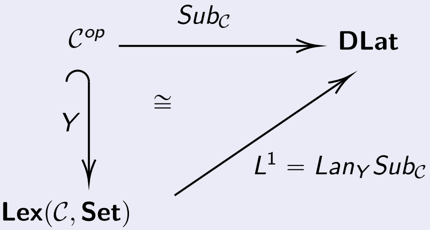

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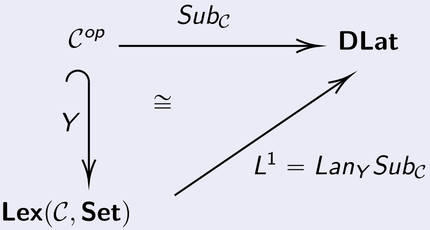

- $L^1(M)$ is the LT-algebra of closed (pos. ex.) formulas with parameters from M.
- L^1 preserves filtered colimits, regular monos, etc.
- Goal: model-theoretic properties of $M : \mathcal{C} \to Sh(B, \tau_{coh})$ vs algebraic properties of $L^1(\Gamma M)$.

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Definition

$F, G : \mathcal{C} \to \mathcal{D}$ lex. A natural transformation $\alpha : F \Rightarrow G$ is elementary if the naturality squares at monos are pullbacks.

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nat. tr.: homomorphism which preserves the formulas in the fragment
elementary nat. tr.: preserves and reflects.

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Definition

$\mathcal{C}$ coh, $\mathcal{E}$ Grothendieck topos, $M : \mathcal{C} \to \mathcal{E}$ coh. M is positively closed if every $M \Rightarrow N$ nat. tr. with N coherent must be elementary.

(global notion)

Definition

$\mathcal{C}$ coh, $\mathcal{E}$ Grothendieck topos, $M : \mathcal{C} \to \mathcal{E}$ coh. M is strongly positively closed if for every $u \hookrightarrow x$ in $\mathcal{C}$:

$$Mx = Mu \cup \bigcup_{v \hookrightarrow x:\ u \cap v = \varnothing} Mv$$

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If $\mathcal{E} = \mathbf{Set}$ then pos. cl. = strongly pos. cl.

Example

X non-discrete extremally disconnected Stone space (i.e. $X = Spec(B)$ where B is a complete Boolean alg., $B \neq 2^I$ for I finite).
Then $int : Closed(X) \to Clopen(X) \subseteq Sh(X)$ is coherent, positively closed, not strongly pos. cl.

$\mathcal{C}$ is coherent, B is a Boolean algebra, $M : \mathcal{C} \to Sh(B, \tau_{coh})$ is coherent.
Write L for $L^1(\Gamma M)$.

Theorem

M is strongly pos. cl. iff L is a Boolean algebra iff $L \cong B$.

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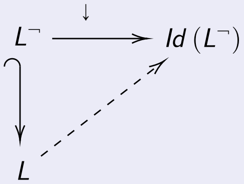
If $Sh(B, \tau_{coh})$ can realize types (e.g. B is complete), then M is pos. cl. iff there is a unique lift



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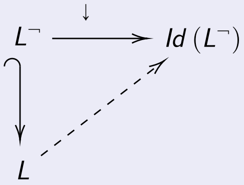

Theorem

If B is complete then M is pos. cl. iff for every $u \hookrightarrow x$ in $\mathcal{C}$ and $s \in Mx(\top)$, there is $b_0 \in B$ with $s|_{b_0} \in Mu(b_0)$ and $\neg b_0 = \bigcup\{b : \exists v : s|_b \in Mv(b) \text{ and } u \cap v = \varnothing\}$.