Compact, Hausdorff and locally compact locales internal in toposes

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Internal frames in presheaves on a small category

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  commutes.

Internal frames associated with a presheaf of frames.

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- On arrows $r\colon b \to a$ in $\mathcal{C}$, $\mathfrak{z}L(r)\colon \mathfrak{z}L(a) \to \mathfrak{z}L(b)$ sends the tuple $(u_f)_{\vartheta_1 f = a}$ to $(v_h)_{\vartheta_1 h = b}$, where $v_h = u_{rh}$.

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- The left adjoint $\Sigma_r^{\mathfrak{z}L}$ to Lr is given by
$$\Sigma_r^{\mathfrak{z}L}(u_f) = (\bigvee_{rf = g} u_f)_g.$$

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- $$\mathrm{idl}_{\widehat{\mathcal{C}}}(S)(a) = \mathfrak{z}(\mathrm{idl} \circ S)(a)$$
- $$\mathrm{idl}_{\widehat{\mathcal{C}}}(S)(a) = \{(I_f) \in \prod_{f\colon b \to a} \mathrm{idl}(S(b)) \mid \forall g\colon c \to b\ (L(g)[I_f] \subseteq I_{fg})\}$$

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- In terms of sections we have $\zeta_a(u) = (\twoheaddownarrow Lf(u))_f$, from which we get, for $I \in \mathrm{idl}(La)$, $\bigvee I = D_a(LfI)_f$ and hence an extra left adjoint $\lambda_a \colon La \to \mathrm{idl}(La)$ can be calculated as

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- $$\lambda_a(u) = \langle \bigcup_f \Sigma_f(\Lambda_a u)_f \rangle.$$
- **Proposition:** The sections of an internally locally compact frame are locally compact.

- **Proposition:** If $L \in \mathbf{Frm}[\mathcal{C}^{op}, \mathbf{Set}]$ is locally compact, then for all the transitions $Lg\colon La \to Lb$, if $w \ll u \in La$, then $Lg(w) \ll Lg(u)$ in Lb.

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- **Proof:** Using lax compatibility of the family $(\Lambda_a u)_f$ and the naturality of Λ and the Beck - Chevalley condition for L, for the pullback along any f $\begin{array}{ccc} \cdot & \xrightarrow{h} & b \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow{f} & a, \end{array}$ we get

$$\begin{aligned} Lg[\langle \Sigma_f (\Lambda_a u)_f \rangle] &= \Sigma_h Lk[(\Lambda_a u)_f] \\ &\subseteq \Sigma_h[(\Lambda_a u)_{fk}] \\ &= \Sigma_h[(\Lambda_a u)_{gh}] \\ \text{naturality of } \Lambda \quad &= \Sigma_h[(\Lambda_b Lgu)_{gh}] \\ &\subseteq \langle \bigcup_h \Sigma_h[(\Lambda_b Lgu)_{gh}] \rangle \\ &= \lambda_b(Lgu) \end{aligned}$$

- Locally compact frames are isomorphic to their lattices of *rounded* ideals (ideals I such that for all $u \in I$ there exists $w \in I$, $u \ll w$):

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- **Theorem:** $L \in \mathbf{Frm}[\mathcal{C}^{op}, \mathbf{Set}]$ is locally compact iff (i) all sections are locally compact, (ii) all transitions $Lg \colon La \to Lb$ preserve $\ll$ and (iii) $L \cong \mathfrak{z}L$. (Using C. Townsend, *Cahiers* LXVI-2, 19–32 (2025).)

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