

Codiscrete cofibrations vs. iterated discrete fibrations for (∞, ℓ) -profunctors and ℓ -congruences

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Exactness in 2-categories (after Bourke–Garner)

\mathfrak{K} an $(\infty, 2)$ -category. Adjunction $\mathcal{C}\mathrm{at}(\mathfrak{K}) \subset \mathfrak{K}^{\Delta^{\mathrm{op}}} \begin{array}{c} \xrightarrow{\mathrm{codesc}} \\ \perp \\ \xleftarrow{\mathrm{ker}} \end{array} \mathfrak{K}^2$

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\mathbb{K} **effective**: restricts to an equivalence $2\text{-}\mathcal{C}\text{ong}(\mathbb{K}) = \mathcal{C}\text{atead}(\mathbb{K}) \simeq \text{eso}(\mathbb{K})$

Categorified congruences

A **2-congruence**, aka **catead**, is an internal category X_{\bullet} whose underlying graph $X_0 \leftarrow X_1 \rightarrow X_0$ is a discrete (i.e. $(\infty, 0)$ -categorical) two-sided fibration

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Remark: eso/ff factorisation system generated by $\mathcal{O} := \{n = \text{obj } \mathfrak{n} \rightarrow \mathfrak{n}\} \subset \mathcal{C}\text{at}^2$
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 $\mathcal{O} \simeq \Delta$, and $\text{codesc} \dashv \text{ker}$ colim/lim weighted by the profunctor $\mathbb{2} \times \Delta^{opop} \rightarrow \mathcal{C}at$

Towards ℓ -categorical exactness

In $\hat{\mathcal{K}}$ an (∞, ℓ) -category ($\ell \in \overline{\mathbb{N}}$), two natural generalisations of congruences:

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Effectivity for ℓ -congruences (Loubaton, Mesiti for $\ell = 3$)

Replace co/limits by lax (*i.e.* Gray-enriched) ones, *e.g.* lax commas

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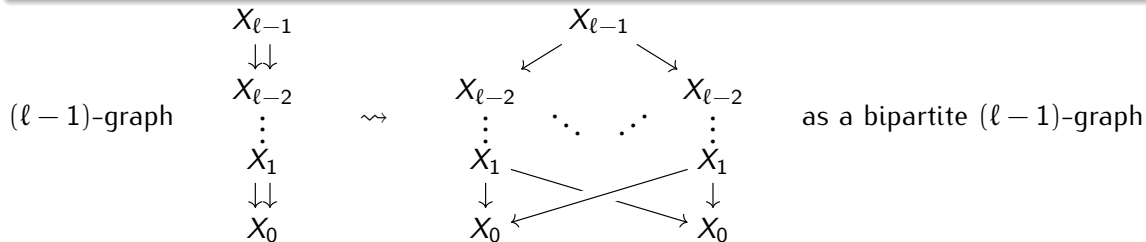
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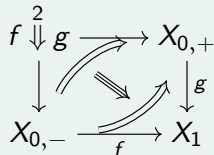
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$$f \Downarrow^k g = \left\{ \begin{array}{ccc} X_{0,-} & & X_{0,+} \\ & \searrow f & \swarrow g \\ & X_1 & \end{array} \right\}, \mathcal{W} = \left(\begin{array}{ccc} 1 & & 1 \\ \lrcorner_{0-} \searrow & & \swarrow \lrcorner_{0+} \\ & \mathcal{D}_k & \end{array} \right)$$



Comparing congruences and cateads

[Moser–Rasekh–Rovelli
Nuiten, Loubaton] $(\ell - 2)$ -cat'l 2-sided fibrations $/A \times B$ are $(\infty, \ell - 1)$ -profunctors $A \rightharpoonup B$

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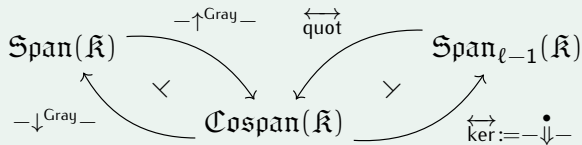
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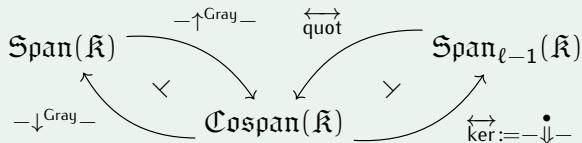


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Adjunctions



Conjecture

For nice $\tilde{\mathcal{K}}$, the LHS adjunction is idempotent, so restricts to $(\ell - 2)\text{-Fib}(\tilde{\mathcal{K}}) \simeq \text{CodisCofib}(\tilde{\mathcal{K}})$

Theorem (WIP, inspired by Bourke's exactness of cateads)

For $\tilde{\mathcal{K}} = (\infty, \ell - 1)\text{-Cat}$, the RHS is idemp., so restricts to $\text{DiscFib}_{\ell-1}(\tilde{\mathcal{K}}) \simeq \text{CodisCofib}(\tilde{\mathcal{K}})$

Application to Mesiti's good classifiers

Fibration classifier (... , Weber, ...)

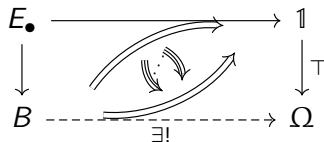
Iterated fibration $\Omega_{*,\bullet} \rightarrow \Omega \times \mathbb{1}$ such that $(-\times_{\Omega} \Omega_{*,\bullet}): \tilde{\mathcal{K}}(-, \Omega) \xrightarrow{\text{ff.}} \mathsf{DiscFib}_\ell(-)$

$$\begin{array}{ccccc} E_{\bullet} & \longrightarrow & \Omega_{*,\bullet} & \longrightarrow & \mathbb{1} \\ \downarrow & \lrcorner & \downarrow & & \\ B & \dashrightarrow & \Omega & & \\ & \exists! & & & \end{array}$$

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Good classifiers (Mesiti)

$\mathbb{T}: \mathbb{1} \rightarrow \Omega$ such that $-\overset{\bullet}{\Downarrow} \mathbb{T}: \mathcal{K}(-, \Omega) \xrightarrow{\text{f.f.}} \mathbf{DiscFib}_\ell(-)$

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Upshot: Universal fibration $\Omega_{*,\bullet} \simeq \text{id}_\Omega \overset{\bullet}{\Downarrow} \top$ is higher pointed objects in Ω

Backup

Computing higher two-sided kernels

Lemma

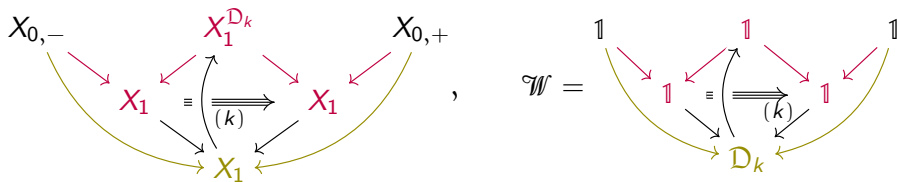
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Proof.



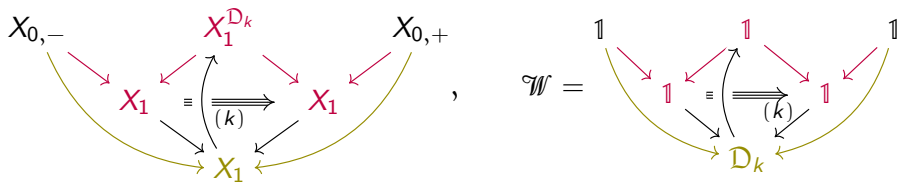
Red diagram is weighted final in the full one, which is right Kan-extended from the green \square

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Corollary

Two-sided kernels are indeed iterated discrete fibrations

Computing two-sided quotients of iterated discrete fibrations

Lemma

For X_\bullet an ℓ -iterated discrete fibration in $\tilde{\mathcal{K}} = (\infty, \ell)\text{-}\mathcal{Cat}$, $\overset{\leftarrow}{\text{quot}}(X_\bullet)_1$ is the “horizontal (∞, ℓ) -category” of the double (∞, ℓ) -category freely generated by X_\bullet .

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Proof.

The right-adjoint to $\overset{\longleftrightarrow}{\text{quot}}(-)_1$ identifies with

$$\tilde{\mathcal{K}} \xrightarrow{Sq} \ell\text{-}\mathcal{Cat}(\tilde{\mathcal{K}}) \rightarrow \ell\text{-}\mathcal{Grph}(\tilde{\mathcal{K}}) \rightarrow \ell\text{-}\mathcal{BipartGrph}(\tilde{\mathcal{K}}) \supset \text{DiscFib}_\ell(\tilde{\mathcal{K}})$$

where Sq constructs the double (∞, ℓ) -categories of strong commutative “squares”

In general $\mathcal{Hor}: \ell\text{-}\mathcal{Cat}((\infty, \ell)\text{-}\mathcal{Cat}) \rightarrow (\infty, \ell)\text{-}\mathcal{Cat}$ is not left-adjoint to Sq , but its restriction to ℓ -cateads is

