Codiscrete cofibrations *vs.* iterated discrete fibrations for (∞, ℓ)-profunctors and ℓ-congruences

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Exactness in 2-categories (after Bourke–Garner)

$\mathfrak{K}$ an $(\infty, 2)$-category. Adjunction
$$\begin{tikzcd} \mathfrak{Cat}(\mathfrak{K}) \subset \mathfrak{K}^{\mathbb{\Delta}^{\mathrm{op}}} \arrow[r, bend left, "\mathrm{codesc}"] \arrow[r, phantom, "\perp"] & \mathfrak{K}^{\mathbb{2}} \arrow[l, bend left, "\mathrm{ker}"] \end{tikzcd}$$

- $\operatorname{codesc} X_\bullet = \big(X_0 \to \mathscr{W} \star X\big)$, where $\mathscr{W} \star X$ **colimit** weighted by $\mathscr{W} : \mathbb{\Delta} \hookrightarrow \mathfrak{Cat}$

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  where $f \downarrow \cdots \downarrow f$ **limit** of $\ulcorner f \urcorner \colon \mathbb{2} \to \mathfrak{K}$ weighted by $(n \to \mathbb{n})$, with $\mathbb{n} \simeq \mathbb{2} \amalg_{\mathbb{1}} \cdots \amalg_{\mathbb{1}} \mathbb{2}$

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$\mathfrak{K}$ **effective**: restricts to an equivalence $2\text{-}\mathfrak{Cong}(\mathfrak{K}) = \mathfrak{Catead}(\mathfrak{K}) \simeq \mathfrak{eso}(\mathfrak{K})$

Categorified congruences

A 2-**congruence**, aka **catead**, is an internal category $X_\bullet$ whose underlying graph $X_0 \leftarrow X_1 \to X_0$ is a discrete (*i.e.* $(\infty,0)$-categorical) two-sided fibration

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Remark: eso/ff factorisation system generated by $\mathfrak{O} \coloneqq \{n = \mathrm{obj}\,\mathbb{n} \to \mathbb{n}\} \subset \mathfrak{Cat}^{\mathbb{2}}$
$\mathfrak{O} \simeq \Delta$

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$\mathbb{O} \simeq \Delta$, and codesc $\dashv$ ker colim/lim weighted by the profunctor $\mathbb{2} \times \Delta^{\mathrm{op\,op}} \to \mathfrak{Cat}$

Towards ℓ-categorical exactness

In $\mathfrak{K}$ an (∞, ℓ)-category ($\ell \in \overline{\mathbb{N}}$), two natural generalisations of congruences:

ℓ-congruences: Internal category $X_\bullet$ whose graph is an $(\ell - 2)$-cat'l two-sided fibration

Effectivity for ℓ-congruences (Loubaton, Mesiti for $\ell = 3$)

Replace co/limits by lax (*i.e.* Gray-enriched) ones, *e.g.* lax commas

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$(\ell-1)$-graph
$$\begin{tikzcd}
X_{\ell-1} \arrow[d, shift left] \arrow[d, shift right] \\
X_{\ell-2} \arrow[d, phantom, "\vdots"] \\
X_1 \arrow[d, shift left] \arrow[d, shift right] \\
X_0
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
& X_{\ell-1} \arrow[dl] \arrow[dr] & \\
X_{\ell-2} \arrow[d, phantom, "\vdots"] & \ddots \quad \iddots & X_{\ell-2} \arrow[d, phantom, "\vdots"] \\
X_1 \arrow[d] \arrow[drr] & & X_1 \arrow[d] \arrow[dll] \\
X_0 & & X_0
\end{tikzcd}$$
as a bipartite $(\ell-1)$-graph

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$$f \overset{k}{\Downarrow} g = \left\{ \begin{tikzcd} X_{0,-} \arrow[dr, "f"'] & & X_{0,+} \arrow[dl, "g"] \\ & X_1 & \end{tikzcd}, \mathscr{W} = \left(\begin{tikzcd} \mathbb{1} \arrow[dr, "\ulcorner 0^- \urcorner"'] & & \mathbb{1} \arrow[dl, "\ulcorner 0^+ \urcorner"] \\ & \mathfrak{D}_k & \end{tikzcd} \right) \right\}$$

$$\begin{tikzcd} f \overset{2}{\Downarrow} g \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rrightarrow"] & X_{0,+} \arrow[d, "g"] \\ X_{0,-} \arrow[r, "f"'] & X_1 \end{tikzcd}$$

Comparing congruences and cateads

[Moser–Rasekh–Rovelli, Nuiten, Loubaton] $(\ell-2)$-cat'l 2-sided fibrations $/A \times B$ are $(\infty, \ell-1)$-profunctors $A \nrightarrow B$

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Street: For $\mathfrak{V}$ enriching monoidal category, $\mathfrak{V}$-profunctors are always encoded by **codiscrete** two-sided **cofibrations** in $\mathfrak{V}$-$\mathfrak{Cat}$

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$$\mathfrak{Cospan}(\mathfrak{K})$$

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Adjunctions

$$\begin{tikzcd}
\mathfrak{Span}(\mathfrak{K}) \arrow[dr, bend left, "-\uparrow^{\mathrm{Gray}}-"] & & \mathfrak{Span}_{\ell-1}(\mathfrak{K}) \arrow[dl, bend right, "\overleftrightarrow{\mathrm{quot}}"'] \\
& \mathfrak{Cospan}(\mathfrak{K}) \arrow[ul, bend left, "-\downarrow^{\mathrm{Gray}}-"] \arrow[ur, bend right, "\overleftrightarrow{\mathrm{ker}} := -\overset{\bullet}{\Downarrow}-"'] &
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Conjecture

For nice $\mathfrak{K}$, the LHS adjunction is idempotent, so restricts to $(\ell-2)$-$\mathfrak{Fib}(\mathfrak{K}) \simeq \mathfrak{CodisCofib}(\mathfrak{K})$

Theorem (WIP, inspired by Bourke's exactness of cateads)

For $\mathfrak{K} = (\infty,\ell-1)$-$\mathfrak{Cat}$, the RHS is idemp., so restricts to $\mathfrak{DiscFib}_{\ell-1}(\mathfrak{K}) \simeq \mathfrak{CodisCofib}(\mathfrak{K})$

Application to Mesiti's good classifiers

Fibration classifier (..., Weber, ...)

Iterated fibration $\Omega_{\star,\bullet} \to \Omega \times \mathbb{1}$ such that $\left(- \underset{\Omega}{\times} \Omega_{\star,\bullet}\right) : \mathfrak{K}(-, \Omega) \xrightarrow{\text{f.f.}} \mathfrak{DiscFib}_{\ell}(-)$

$$\begin{tikzcd}
E_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_{\star,\bullet} \arrow[r] \arrow[d] & \mathbb{1} \\
B \arrow[r, dashed, "\exists!"'] & \Omega &
\end{tikzcd}$$

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Good classifiers (Mesiti)

$\top \colon \mathbb{1} \to \Omega$ such that $- \overset{\bullet}{\Downarrow} \top \colon \mathfrak{K}(-, \Omega) \xrightarrow{\text{f.f.}} \mathfrak{DiscFib}_\ell(-)$

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Upshot: Universal fibration $\Omega_{\star,\bullet} \simeq \mathrm{id}_\Omega \overset{\bullet}{\Downarrow} \top$ is higher pointed objects in Ω

Backup

Computing higher two-sided kernels

Lemma

For $X_{0,-} \xrightarrow{f} X_1 \xleftarrow{g} X_{0,+}$, we have $f \overset{k}{\Downarrow} g \simeq X_{0,-} \underset{X_1}{\times} X_1^{\mathcal{D}_k} \underset{X_1}{\times} X_{0,+}$

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Proof.

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& X_1 \arrow[dr] \arrow[rr, phantom, "\equiv \; \Rrightarrow_{(k)}"] & & X_1 \arrow[dl] & \\
& & X_1 \arrow[uu, bend left] & &
\end{tikzcd}
\quad , \quad
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Red diagram is weighted final in the full one, which is right Kan-extended from the green □

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Corollary

Two-sided kernels are indeed iterated discrete fibrations

Computing two-sided quotients of iterated discrete fibrations

Lemma

For $X_\bullet$ an ℓ-iterated discrete fibration in $\mathfrak{K} = (\infty,\ell)\text{-}\mathfrak{Cat}$, $\overleftrightarrow{\mathrm{quot}}(X_\bullet)_1$ is the "horizontal (∞,ℓ)-category" of the double (∞,ℓ)-category freely generated by $X_\bullet$

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Proof.

The right-adjoint to $\overleftrightarrow{\mathrm{quot}}(-)_1$ identifies with

$$\mathfrak{K} \xrightarrow{\mathscr{S}q} \ell\text{-}\mathfrak{Cat}(\mathfrak{K}) \to \ell\text{-}\mathfrak{Grph}(\mathfrak{K}) \to \ell\text{-}\mathfrak{BipartGrph}(\mathfrak{K}) \supset \mathfrak{DiscFib}_\ell(\mathfrak{K})$$

where $\mathscr{S}q$ constructs the double (∞, ℓ)-categories of strong commutative "squares"

In general $\mathscr{H}or\colon \ell\text{-}\mathfrak{Cat}\big((\infty, \ell)\text{-}\mathfrak{Cat}\big) \to (\infty, \ell)\text{-}\mathfrak{Cat}$ is not left-adjoint to $\mathscr{S}q$,
but its restriction to ℓ-cateads is □