Linearly Distributive Fox Theorem

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Main Result

Theorem (Fox Theorem)

$$\mathrm{inc} \dashv \mathrm{C}[-] : \textbf{CART} \to \textbf{SMC}$$

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Theorem (Linearly Distributive Fox Theorem)

$$\mathrm{inc} \dashv B[-] : \textbf{CLDC} \to \textbf{SMLDC}$$

- $\Rightarrow$ **SMLDC**: 2-category of symmetric medial LDCs
- $\Rightarrow$ $B[\mathbb{X}]$: category of bicommutative medial bimonoids

Girard introduced a sub-structural logic in 1987 [10]:

Linear Logic

	Multiplicative	Additive	Exponential
Conjunction	$\otimes$, **1**	$\&$, $\top$	!
Disjunction	$⅋$, $\bot$	$\oplus$, **0**	?
	Implication		Negation
Linear	$\multimap$		$(-)^{\perp}$

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Categorical semantics were investigated by Seely [15] and it was shown that *multiplicative linear logic (MLL) with negation* corresponds to Barr's $*$**-autonomous categories** [2]:

- a SMC $(\mathbb{X}, \otimes, \mathbf{1})$ with
- a full and faithful functor $(-)^{\perp} : \mathbb{X}^{op} \to \mathbb{X}$ such that

$$\mathbb{X}(A \otimes B, C^{\perp}) \cong \mathbb{X}(A, (B \otimes C)^{\perp})$$

$\Rightarrow$ Multiplicative conjunction and linear negation are taken as the primitive categorical notions.

Linearly distributive categories

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- a category $(\mathbb{X}, ;, 1_A)$,
- a **tensor** monoidal structure $(\mathbb{X}, \otimes, \top)$,
- a **par** monoidal structure $(\mathbb{X}, \oplus, \bot)$, and
- left and right **linear distributivity** natural transformations

$$\delta^R_{A,B,C} \colon (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

$$\delta^L_{A,B,C} \colon A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

satisfying coherence conditions.

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Remark. Notational conflict

	Tensor	Par	With	Plus
Cockett+Seely	$\otimes. \top$	$\oplus, \bot$	$\times, 1$	$+, 0$
Girard	$\otimes, \mathbf{1}$	$\parr, \bot$	$\&, \top$	$\oplus, \mathbf{0}$

Definition (Cockett, Seely [6])

A LDC $\mathbb{X}$ is **symmetric**, or a SLDC, if

- $(\mathbb{X}, \otimes, \top)$ is symmetric with $\otimes$-braiding

$$\sigma_{\otimes A,B} : A \otimes B \to B \otimes A$$

- $(\mathbb{X}, \oplus, \bot)$ is symmetric with $\oplus$-braiding

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such that

$$\begin{array}{ccc}
(A \oplus B) \otimes C & \xrightarrow{\delta^R_{A,B,C}} & A \oplus (B \otimes C) \\
\downarrow{\scriptstyle \sigma_{\otimes A\oplus B,C}} & & \uparrow{\scriptstyle \sigma_{\oplus B\otimes C,A}} \\
C \otimes (A \oplus B) & & (B \otimes C) \oplus A \\
\downarrow{\scriptstyle 1_C \otimes \sigma_{\oplus A,B}} & & \uparrow{\scriptstyle \sigma_{\otimes C,B} \oplus 1_A} \\
C \otimes (B \oplus A) & \xrightarrow[\delta^L_{C,B,A}]{} & (C \otimes B) \oplus A
\end{array}$$

Mix LDCs

$$\frac{\Gamma \vdash \Delta \qquad \Theta \vdash \Psi}{\Gamma, \Theta \vdash \Delta, \Psi} \text{(MIX)}$$

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Definition (Cockett, Seely [5])

A LDC $\mathbb{X}$ is **mix** if there is a map $m : \bot \to \top$ such that

$$\begin{array}{ccccc}
A \otimes B & \xrightarrow{1_A \otimes {u^L_{\oplus}}_B^{-1}} & A \otimes (\bot \oplus B) & \xrightarrow{1_A \otimes (m \oplus 1_B)} & A \otimes (\top \oplus B) \\
\downarrow {u^R_{\oplus}}_A^{-1} \otimes 1_B & & & & \downarrow \delta^L_{A,\top,B} \\
(A \oplus \bot) \otimes B & & & & (A \otimes \top) \oplus B \\
\downarrow (1_A \oplus m) \otimes 1_B & & & & \downarrow {u^R_{\otimes}}_A^{-1} \oplus 1_B \\
(A \oplus \top) \otimes B & \xrightarrow[\delta^R_{A,\top,B}]{} & A \oplus (\top \otimes B) & \xrightarrow[1_A \oplus {u^L_{\otimes}}_B^{-1}]{} & A \oplus B
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in which case there is a natural transformation

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\downarrow {\scriptstyle (1_A \oplus m) \otimes 1_B} & & & & \downarrow {\scriptstyle {u^R_\otimes}_A^{-1} \oplus 1_B} \\
(A \oplus \top) \otimes B & \xrightarrow[\delta^R_{A,\top,B}]{} & A \oplus (\top \otimes B) & \xrightarrow[1_A \oplus {u^L_\otimes}_B^{-1}]{} & A \oplus B
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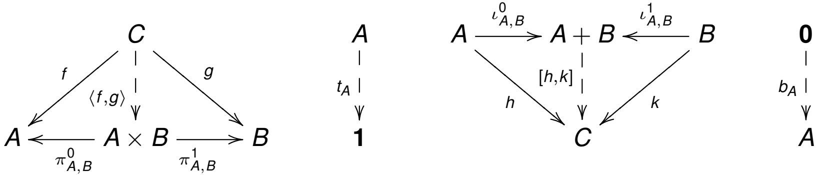
A LDC is **isomix** if it is mix and $m : \bot \to \top$ is an isomorphism.

Definition (Cockett, Seely [6])

A **cartesian linearly distributive category**, or CLDC, $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$ is a SLDC whose

- tensor structure is cartesian - the categorical product $\times$ with the terminal object $\mathbf{1}$, and
- par structure is cocartesian - the categorical coproduct $+$ with the initial object $\mathbf{0}$.



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$$\begin{array}{ccc} (A+B)\times C & \xrightarrow{\psi_{A,B}\times 1_C} & (A\times B)\times C \\ {\scriptstyle \delta^R_{A,B,C}}\downarrow & & \downarrow{\scriptstyle \alpha^{\times}_{A,B,C}} \\ A+(B\times C) & \xleftarrow[\psi^{-1}_{A,B\times C}]{} & A\times(B\times C) \end{array} \qquad \begin{array}{ccc} A\times(B+C) & \xrightarrow{1_A\times\psi_{B,C}} & A\times(B\times C) \\ {\scriptstyle \delta^L_{A,B,C}}\downarrow & & \downarrow{\scriptstyle {\alpha^{\times}}^{-1}_{A,B,C}} \\ (A\times B)+C & \xleftarrow[\psi^{-1}_{A\times B,C}]{} & (A\times B)\times C \end{array}$$

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3. Product of CLDCs

Motivation

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(2)

Direct investigation of properties and examples of CLDCs

⇒ Cartesian Linearly Distributive Categories: Revisited - jww JS Lemay
(*to appear soon*)

Fox's theorem

Consider a SMC $(\mathcal{X}, \oslash, I)$. Note the *canonical flip*:

$$\tau^{\oslash}_{A,B,C,D} : (A \oslash B) \oslash (C \oslash D) \xrightarrow{\sim} (A \oslash C) \oslash (B \oslash D)$$

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Proposition (Fox [8])

Given cocommutative comonoids $\langle A, \Delta_A, e_A \rangle$ and $\langle B, \Delta_B, e_B \rangle$, then $\langle A \oslash B, \Delta_{A \oslash B}, e_{A \oslash B} \rangle$ as defined below is a cocommutative comonoid.

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- *it is a cartesian category.*

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Corollary

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Corollary (Heunen, Vicary [11])

A SMC $\mathcal{X}$ *is cartesian if and only if there are natural transformations*

$$e_A : A \to I \qquad \Delta_A : A \to A \oslash A$$

such that $\langle A, \Delta_A, e_A \rangle$ *is cocommutative comonoid and*

$$e_{A \oslash B} = (e_A \oslash e_B); \rho_I^{-1} \qquad e_I = 1_I$$

$$\Delta_{A \oslash B} = (\Delta_A \oslash \Delta_B); \tau^{\oslash}_{A,A,B,B} \qquad \Delta_I = \rho_I.$$

Back to CLDCs

By Fox's theorem and its dual applied to CLDCs, we can show that there are natural transformations

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If we consider any SLDC $\mathbb{X}$ and try forming the category of such quintuples $\langle A, \Delta_A, e_A, \nabla_A, u_A \rangle$, we quickly realize we *cannot define* a tensor or par product:

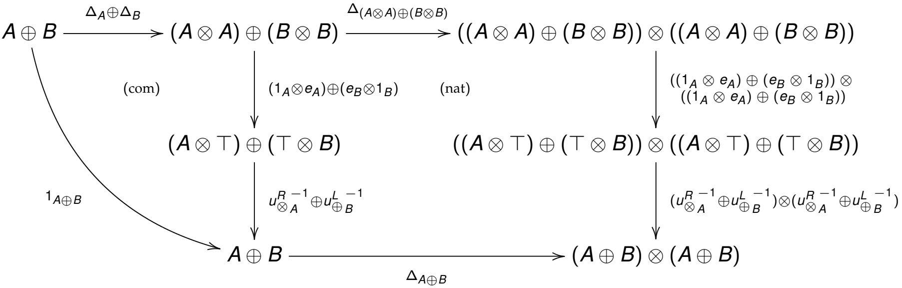
$$e_{A \oplus B} : A \oplus B \xrightarrow{e_A \oplus e_B} \top \oplus \top \xrightarrow{?} \top$$

$$\Delta_{A \oplus B} : A \oplus B \xrightarrow{\Delta_A \oplus \Delta_B} (A \otimes A) \oplus (B \otimes B) \xrightarrow{?} (A \oplus B) \otimes (A \oplus B)$$

$$u_{A \otimes B} : \bot \xrightarrow{?} \bot \otimes \bot \xrightarrow{u_A \otimes u_B} A \otimes B$$

$$\nabla_{A \otimes B} : (A \otimes B) \oplus (A \otimes B) \xrightarrow{?} (A \oplus A) \otimes (B \oplus B) \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B$$

Moreover, consider a CLDC once more, then $\Delta_{A\oplus B}$:



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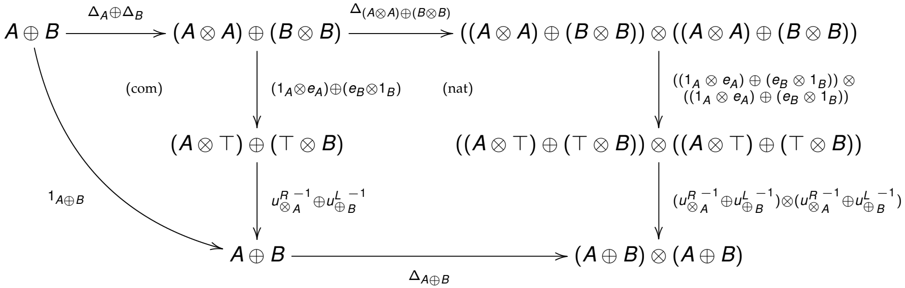

$$\Delta_{A\oplus B} = A \oplus B \xrightarrow{\Delta_A \oplus \Delta_B} (A \otimes A) \oplus (B \otimes B) \xrightarrow{\mu_{A,A,B,B}} (A \oplus B) \otimes (A \oplus B)$$

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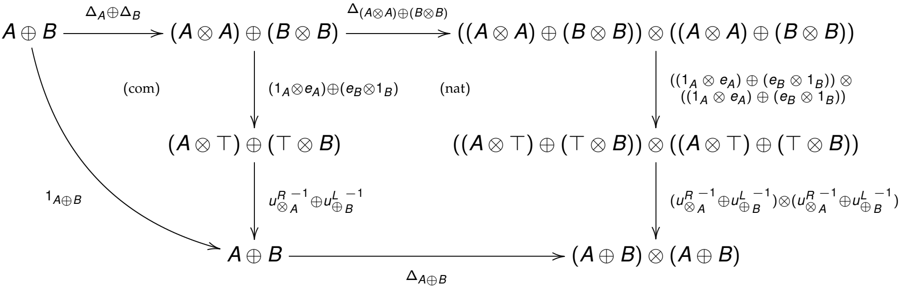

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We need to start with a SLDC $\mathbb{X}$ with such arrows.

Medial rule

In logic, $(A \otimes B) \oplus (C \otimes D) \rightarrow (A \oplus C) \otimes (B \oplus D)$ is known as the **medial rule**.

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$\Rightarrow$ In every case, the medial rule is considered for the same reason it appears currently.

Duoidal categories

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Definition (Aguiar, Mahajan [1])

A **duoidal category** $(\mathcal{X}, \diamond, I, \star, J)$ is category $\mathcal{X}$ with two monoidal structures $(\mathcal{X}, \diamond, I)$ and $(\mathcal{X}, \star, J)$ equipped with morphisms

$$\Delta_I : I \to I \star I \qquad \mu_J : J \diamond J \to J \qquad \iota : I \to J$$

and an **interchange** natural transformation

$$\zeta_{A,B,C,D} : (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

satisfying some coherence conditions.

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- *$\bot$-contraction*, *$\top$-cocontraction* and *nullary mix* morphisms,

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$$\delta^R_{A,B,C} : (A \oplus B) \otimes C \to A \oplus (B \otimes C) \qquad \delta^L_{A,B,C} : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

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such that

- $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is a *mix LDC*,
- $(\mathbb{X}, \oplus, \bot, \otimes, \top)$ is a *duoidal category*, and
- the medial maps interact coherently with the linear distributivities.

Example

1. Braided monoidal categories $(\mathcal{X}, \oslash, I, \oslash, I)$

$$\nabla_I = \lambda_I^{-1} : I \oslash I \to I \qquad \Delta_I = \rho_I : I \to I \oslash I \qquad m = 1_I : I \to I$$

$$\mu_{A,B,C,D} = \tau^{\oslash}_{A,B,C,D} : (A \oslash B) \oslash (C \oslash D) \to (A \oslash C) \oslash (B \oslash D)$$

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2. Cartesian linearly distributive category $(\mathbb{X}, \times, \mathbf{1}, +, \mathbf{0})$

$$\nabla_{\mathbf{1}} = t_{\mathbf{1}+\mathbf{1}} : \mathbf{1} + \mathbf{1} \to \mathbf{1} \quad \Delta_{\mathbf{0}} = b_{\mathbf{0}\times\mathbf{0}} : \mathbf{0} \to \mathbf{0} \times \mathbf{0} \quad m = t_{\mathbf{0}} = b_{\mathbf{1}} : \mathbf{0} \to \mathbf{1}$$
$$\mu_{A,B,C,D} = [\iota^0_{A,C} \times \iota^0_{B,D}, \iota^1_{A,C} \times \iota^1_{B,D}] = \langle \pi^0_{A,B} + \pi^0_{C,D}, \pi^1_{A,B} + \pi^1_{C,D} \rangle :$$
$$(A \times B) + (C \times D) \to (A + C) \times (B + D)$$
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3. Category of $\mathbb{P}$-coherences and $\mathbb{P}$-coherence maps $\mathbb{P}$-Coh, in the sense of Lamarche [13], for some posetal symmetric MLDC $\mathbb{P}$ (e.g. a bounded distributive lattice)

Definition

A **symmetric MLDC**, or SMLDC, $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is a MLDC with braidings $\sigma_\otimes$ and $\sigma_\oplus$ such that $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ is a SLDC, and $(\mathbb{X}, \oplus, \bot, \otimes, \top)$ is a symmetric duoidal category.

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Given a SMLDC $(\mathbb{X}, \otimes, \top, \oplus, \bot)$*, the following are equivalent:*

1. *the LDC is compact and the duoidal structure is strong,*
   - *it is isomix,*
   - *the mix maps* $\mathrm{mix}_{A,B} : A \otimes B \to A \oplus B$ *are isomorphisms,*
   - *the linear distributivities are associators (modulo the mix maps),*
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2. *the linear distributivities are isomorphisms, and*
3. *the LDC is isomix.*

Medial bimonoids

Definition

Let $\mathbb{X}$ be a SMLDC. A **bicommutative medial bimonoid** in $\mathbb{X}$ is a quintuple $\langle A, \Delta_A, u_A, \nabla_A, e_A \rangle$ consisting of an object A and

$$\Delta_A : A \to A \otimes A \quad e_A : A \to \top \quad \nabla_A : A \oplus A \to A \quad u_A : \bot \to A$$

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in $\mathbb{X}$ such that $\langle A, \Delta_A, e_A \rangle$ is a cocommutative $\otimes$-comonoid and $\langle A, \nabla_A, u_A \rangle$ is a commutative $\oplus$-monoid, satisfying

$$\begin{array}{ccccc}
A \oplus A & \xrightarrow{\nabla_A} & A & \xrightarrow{\Delta_A} & A \otimes A \\
{\scriptstyle \Delta_A \oplus \Delta_A}\downarrow & & & & \uparrow{\scriptstyle \nabla_A \otimes \nabla_A} \\
(A \otimes A) \oplus (A \otimes A) & & \xrightarrow[\mu_{A,A,A,A}]{} & & (A \oplus A) \otimes (A \oplus A)
\end{array}
\qquad
\begin{array}{ccc}
\bot & \xrightarrow{m} & \top \\
{\scriptstyle u_A}\downarrow & \nearrow{\scriptstyle e_A} & \\
A & &
\end{array}$$

$$\begin{array}{ccc}
A \oplus A & \xrightarrow{\nabla_A} & A \\
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\top \oplus \top & \xrightarrow[\nabla_\top]{} & \top
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\bot & \xrightarrow{u_A} & A \\
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Alternatively, it is a bicommutative duoidal bimonoid in the duoidal structure of $\mathbb{X}$.

Medial bimonoids

Proposition

Given two bicommutative medial bimonoids $\langle A, \Delta_A, e_A, \nabla_A, u_A \rangle$ *and* $\langle B, \Delta_B, e_B, \nabla_B, u_B \rangle$ *in* $\mathbb{X}$*, then* $\langle A \otimes B, \Delta_{A\otimes B}, e_{A\otimes B}, \nabla_{A\otimes B}, u_{A\otimes B} \rangle$ *defined by*

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$$\nabla_{A\otimes B} = (A \otimes B) \oplus (A \otimes B) \xrightarrow{\mu_{A,B,A,B}} (A \oplus A) \otimes (B \oplus B) \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B$$

$$e_{A\otimes B} = A \otimes B \xrightarrow{e_A \otimes e_B} \top \otimes \top \xrightarrow{\sim} \top \qquad u_{A\otimes B} = \bot \xrightarrow{\Delta_\bot} \bot \otimes \bot \xrightarrow{u_A \otimes u_B} A \otimes B$$

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$$\Delta_{A\oplus B} = A \oplus B \xrightarrow{\Delta_A \oplus \Delta_B} (A \otimes A) \oplus (B \otimes B) \xrightarrow{\mu_{A,A,B,B}} (A \oplus B) \otimes (A \oplus B)$$

$$\nabla_{A\oplus B} = (A \oplus B) \oplus (A \oplus B) \xrightarrow{\tau^{\oplus}_{A,B,A,B}} (A \oplus A) \oplus (B \oplus B) \xrightarrow{\nabla_A \oplus \nabla_B} A \oplus B$$

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are bicommutative medial bimonoids.

LDC of medial bimonoids

Definition

Let $\mathbb{X}$ be a SMLDC. A **medial bimonoid morphism** is a morphism $f : A \to B$ in $\mathbb{X}$ such that

- $f : \langle A, \Delta_A, e_A \rangle \to \langle B, \Delta_B, e_B \rangle$ is a $\otimes$-comonoid morphism, and
- $f : \langle A, \nabla_A, u_A \rangle \to \langle B, \nabla_B, u_B \rangle$ is a $\oplus$-monoid morphism.

Define $\mathrm{B}[\mathbb{X}]$ to be the category of bicommutative medial bimonoids and bimonoid morphisms in $\mathbb{X}$.

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Proposition

$\mathrm{B}[\mathbb{X}]$ *is a CLDC.*

Definition (Cockett, Seely [7])

A (bilax) **linear functor** $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ consists of:

Linear functors

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$$v^R_{\otimes A,B} \colon F_\otimes(A \oplus B) \to F_\oplus(A) \oplus F_\otimes(B)$$
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subject to various coherence conditions.

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subject to various coherence conditions.

Remark. There is a notion of Frobenius linear functor which amounts to a lax $\otimes$-monoidal/colax $\oplus$-monoidal functor which interacts coherently with the linear distributivities [4].

Definition (Cockett, Seely [7])

If $\mathbb{X}$ and $\mathbb{Y}$ are SLDCs, then a linear functor $F = (F_\otimes, F_\oplus)$ is **symmetric** if $F_\otimes$ and $F_\oplus$ are symmetric, and

$$
\begin{array}{ccc}
F_\otimes(A \oplus B) & \xrightarrow{\nu^L_\otimes} & F_\otimes(A) \oplus F_\oplus(B) \\
{\scriptstyle F_\otimes(\sigma_\oplus)}\downarrow & & \uparrow{\scriptstyle \sigma_\oplus} \\
F_\otimes(B \oplus A) & \xrightarrow[\nu^R_\otimes]{} & F_\oplus(B) \oplus F_\otimes(A)
\end{array}
\qquad
\begin{array}{ccc}
F_\oplus(A) \otimes F_\otimes(B) & \xrightarrow{\nu^L_\oplus} & F_\oplus(A \otimes B) \\
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Definition

A **strong linear functor** is a linear functor $F = (F_\otimes, F_\oplus) \colon \mathbb{X} \to \mathbb{Y}$ where $F_\otimes$ and $F_\oplus$ are monoidal functors.

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Definition

A **strong linear functor** is a linear functor $F = (F_\otimes, F_\oplus) \colon \mathbb{X} \to \mathbb{Y}$ where $F_\otimes$ and $F_\oplus$ are monoidal functors. A strong symmetric linear functor between CLDCs is known as a **cartesian linear functor**.

Duoidal functors

Definition (Aguiar, Mahajan [1])

A **bilax duoidal functor** $(F, p_I, p_\diamond, q_J, q_\star)\colon \mathcal{X} \to \mathcal{Y}$ is a functor $F\colon \mathcal{X} \to \mathcal{Y}$ such that

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Proposition (Aguiar, Mahajan [1])

A bilax duoidal functor preserves bimonoids and morphisms between bimonoids.

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- a functor $F_\oplus : \mathbb{X} \to \mathbb{Y}$, equipped with

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such that

- $F = (F_\otimes, F_\oplus)$ is a *symmetric strong linear functor*,
- $(F_\otimes, m_\bot, m_\oplus, m_\top^{-1}, m_\otimes^{-1})$ is a *bilax duoidal functor*,
- $(F_\oplus, n_\bot^{-1}, n_\oplus^{-1}, n_\top, n_\oplus)$ is a *bilax duoidal functor*, and

Medial linear functors

Definition

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$$
\begin{array}{ccccc}
F_\oplus(\perp) & \xrightarrow{n_\perp} & \perp & \xrightarrow{\Delta_\perp} & \perp \otimes \perp \\
\downarrow{\scriptstyle F_\oplus(\Delta_\perp)} & & & & \downarrow{\scriptstyle m_\perp \otimes n_\perp^{-1}} \\
F_\oplus(\perp \otimes \perp) & & \xleftarrow[\nu^R_{\oplus \perp,\perp}]{} & & F_\otimes(\perp) \otimes F_\oplus(\perp)
\end{array}
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\begin{array}{ccccc}
F_\otimes(\top \oplus \top) & & \xrightarrow{\nu^R_{\otimes \top,\top}} & & F_\oplus(\top) \oplus F_\otimes(\top) \\
\downarrow{\scriptstyle F_\oplus(\nabla_\top)} & & & & \downarrow{\scriptstyle n_\top \oplus m_\top^{-1}} \\
F_\otimes(\top) & \xleftarrow[m_\top]{} & \top & \xleftarrow[\nabla_\top]{} & \top \oplus \top
\end{array}
$$

$$
\begin{array}{ccc}
F_\otimes((A \otimes B) \oplus (C \otimes D)) & \xrightarrow{F_\otimes(\mu)} & F_\otimes((A \oplus C) \otimes (B \oplus D)) \\
\downarrow{\scriptstyle \nu^R_\otimes} & & \downarrow{\scriptstyle m_\otimes^{-1}} \\
F_\oplus(A \otimes B) \oplus F_\otimes(C \otimes D) & & F_\otimes(A \oplus C) \otimes F_\otimes(B \oplus D) \\
\downarrow{\scriptstyle n_\otimes \oplus m_\otimes^{-1}} & & \downarrow{\scriptstyle \nu^R_\otimes \otimes \nu^R_\otimes} \\
(F_\oplus(A) \otimes F_\oplus(B)) \oplus (F_\otimes(C) \otimes F_\otimes(D)) & \xrightarrow[\mu]{} & (F_\oplus(A) \oplus F_\otimes(C)) \otimes (F_\oplus(B) \oplus F_\otimes(D))
\end{array}
$$

Medial linear functors

Lemma

Consider a symmetric medial linear functor $F = (F_\otimes, F_\oplus) : \mathbb{X} \to \mathbb{Y}$ *between SMLDCs, then it canonically extends to a cartesian linear functor* $\mathrm{B}[F] = (\mathrm{B}[F]_\otimes, \mathrm{B}[F]_\oplus) : \mathrm{B}[\mathbb{X}] \to \mathrm{B}[\mathbb{Y}]$*, where*

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$$\Delta_{F_\otimes(A)} = F_\otimes(A) \xrightarrow{F_\otimes(\Delta_A)} F_\otimes(A \otimes A) \xrightarrow{{m_\otimes}_{A,A}^{-1}} F_\otimes(A) \otimes F_\otimes(A)$$

$$e_{F_\otimes(A)} = F_\otimes(A) \xrightarrow{F_\otimes(e_A)} F_\otimes(\top) \xrightarrow{m_\top^{-1}} \top$$

$$\nabla_{F_\otimes(A)} = F_\otimes(A) \oplus F_\otimes(A) \xrightarrow{{m_\oplus}_{A,A}} F_\otimes(A \oplus A) \xrightarrow{F_\otimes(\nabla_A)} F_\otimes(A)$$

$$u_{F_\otimes(A)} = \bot \xrightarrow{m_\bot} F_\otimes(\bot) \xrightarrow{F_\otimes(u_A)} F_\otimes(A)$$

and $\mathrm{B}[F]_\oplus$ *is defined similarly.*

Main Result

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- *CLDCs, cartesian linear functors and linear transformations form a 2-category* **CLDC**.

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Theorem (Linearly Distributive Fox Theorem)

$\mathrm{inc} \dashv \mathrm{B}[-] : \textbf{CLDC} \to \textbf{SMLDC}$.

Corollary

A SMLDC is cartesian if and only if it is isomorphic to its category of bicommutative medial bimonoids and medial bimonoid morphisms.

THANK YOU FOR LISTENING

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https://sites.google.com/view/rosekudzmanblais/home

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Medial linear transformation

Definition

Let $F, G : \mathbb{X} \to \mathbb{Y}$ be symmetric medial linear functors. A **medial linear transformation** $\alpha = (\alpha_\otimes, \alpha_\oplus) : F \Rightarrow G$ consists of:

- a natural transformation $\alpha_\otimes : F_\otimes \Rightarrow G_\otimes$ such that
  - $\alpha_\otimes : (F_\otimes, m_\top^F, m_\otimes^F) \Rightarrow (G_\otimes, m_\top^G, m_\otimes^G)$ is a monoidal transformation,
  - $\alpha_\otimes : (F_\otimes, m_\perp^F, m_\oplus^F) \Rightarrow (G_\otimes, m_\perp^G, m_\oplus^G)$ is a monoidal transformation,
- a natural transformation $\alpha_\oplus : G_\oplus \Rightarrow F_\oplus$ such that
  - $\alpha_\oplus : (G_\oplus, n_\perp^G, n_\oplus^G) \Rightarrow (F_\oplus, n_\perp^F, n_\oplus^F)$ is a comonoidal transformation,
  - $\alpha_\oplus : (G_\oplus, {n_\top}^G, {n_\otimes}^G) \Rightarrow (F_\oplus, n_\top^F, {n_\otimes}^F)$ is a comonoidal transformation,

such that $\alpha = (\alpha_\otimes, \alpha_\oplus)$ is a linear transformation.

Remark. Conditions above are equivalent to

$$\alpha_\otimes : (F_\otimes, m_\perp^F, {m_\oplus}^F, {m_\top^{-1}}^F, {m_\otimes^{-1}}^F) \Rightarrow (G_\otimes, m_\perp^G, {m_\oplus}^G, {m_\top^{-1}}^G, {m_\otimes^{-1}}^G)$$

$$\alpha_\oplus : (G_\oplus, {n_\perp^{-1}}^G, {n_\oplus^{-1}}^G, {n_\oplus}^G, {n_\top}^G) \Rightarrow (F_\oplus, {n_\perp^{-1}}^F, {n_\oplus^{-1}}^F, {n_\oplus}^F, {n_\top}^F)$$

being bilax duoidal transformations.