MACQUARIE UNIVERSITY

MARCELLO LANFRANCHI

# THE FORMAL THEORY OF VECTOR FIELDS FOR TANGENTADS

ARXIV 2503.18354

# DIFFERENT FLAVOURS & CONSTRUCTIONS

MOTIVATION

## DIFFERENT FLAVOURS

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## DIFFERENT FLAVOURS

### TANGENT CATEGORIES

Categorical context for differential geometry

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Fibrations on tangent categories

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Tangent categories with partial maps

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# DIFFERENT FLAVOURS & CONSTRUCTIONS

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MOTIVATION

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Lie algebroids

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DIFFERENT FLAVOURS & CONSTRUCTIONS

**CAN WE FIND A FORMAL THEORY FOR ALL THESE DIFFERENT FLAVOURS?**

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DIFFERENT FLAVOURS & CONSTRUCTIONS

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**TANGENTADS**

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Tangentads in the 2-category of categories

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Tangentads in the 2-category of categories

### TANGENT FIBRATIONS

Tangentads in the 2-category of fibrations

MOTIVATION

# THE FORMAL THEORY OF TANGENTADS

| DIFFERENT FLAVOURS | TANGENTADS |
|---|---|
| **TANGENT CATEGORIES** | **TANGENT CATEGORIES** |
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MOTIVATION

# THE FORMAL THEORY OF TANGENTADS

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| Tangent categories with partial maps | These are not tangentads! |

MOTIVATION

# THE FORMAL THEORY OF TANGENTADS

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| **SPLIT RESTRICTION TANGENT CATS** | **SPLIT RESTRICTION TANGENT CATS** |
| Tangent categories with partial maps | Tangentads in the 2-category of split restriction cats |

# THE FORMAL THEORY OF TANGENTADS

## MOTIVATION

**WHAT ABOUT THE CONSTRUCTIONS?**

THE FORMAL THEORY OF TANGENTADS

MOTIVATION

WHAT ABOUT THE CONSTRUCTIONS?

CAN WE FORMALLY DEFINE THE NOTION OF VECTOR FIELDS?

THE PLAN

# THE FORMAL THEORY OF TANGENTADS

Tangent categories

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THE PLAN

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# THE FORMAL THEORY OF TANGENTADS

Tangent categories

Tangentads

The universal property
of vector fields

# THE FORMAL THEORY OF TANGENTADS

THE PLAN

Tangent categories

Tangentads

The universal property
of vector fields

Some constructions

CHAPTER 1

# TANGENT CATEGORIES

# TANGENT CATEGORIES

ROSICKÝ 1984

COCKETT CRUTTWELL 2014

DEFINITION

Category $\mathbb{X}$

# TANGENT CATEGORIES

ROSICKÝ 1984 COCKETT CRUTTWELL 2014

DEFINITION

Category $\mathbb{X}$

Tangent bundle functor

$T : \mathbb{X} \longrightarrow \mathbb{X}$

# TANGENT CATEGORIES

ROSICKÝ 1984 COCKETT CRUTTWELL 2014

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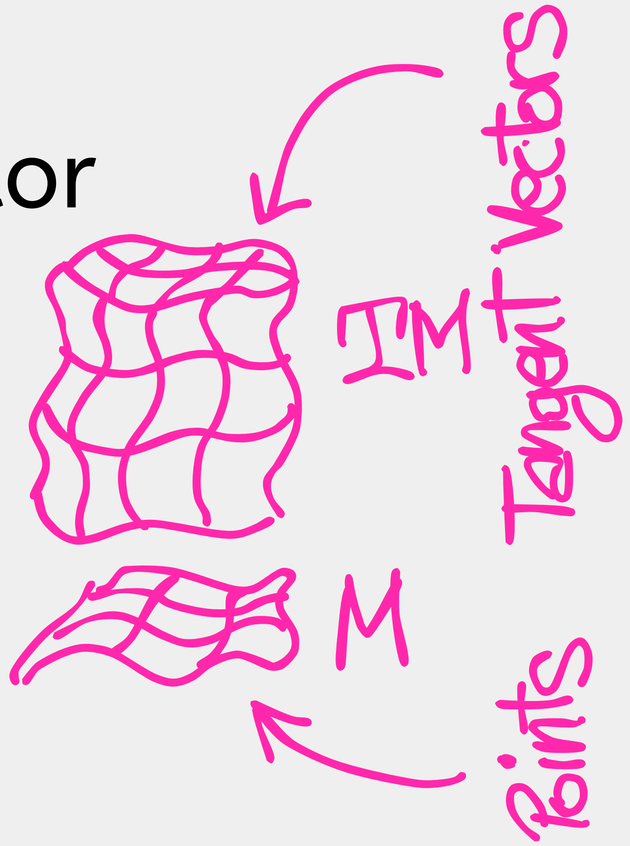
$\mathrm{T} : \mathbb{X} \to \mathbb{X}$



# TANGENT CATEGORIES

ROSICKÝ 1984 COCKETT CRUTTWELL 2014

DEFINITION

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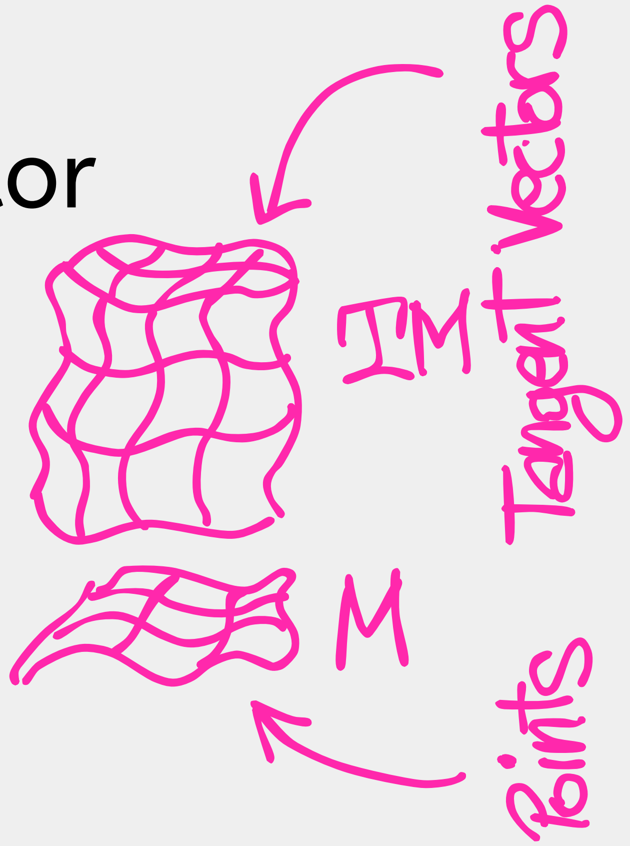
$$\mathrm{T} : \mathbb{X} \longrightarrow \mathbb{X}$$

Projection

$$p_M : \mathrm{T}M \longrightarrow M$$



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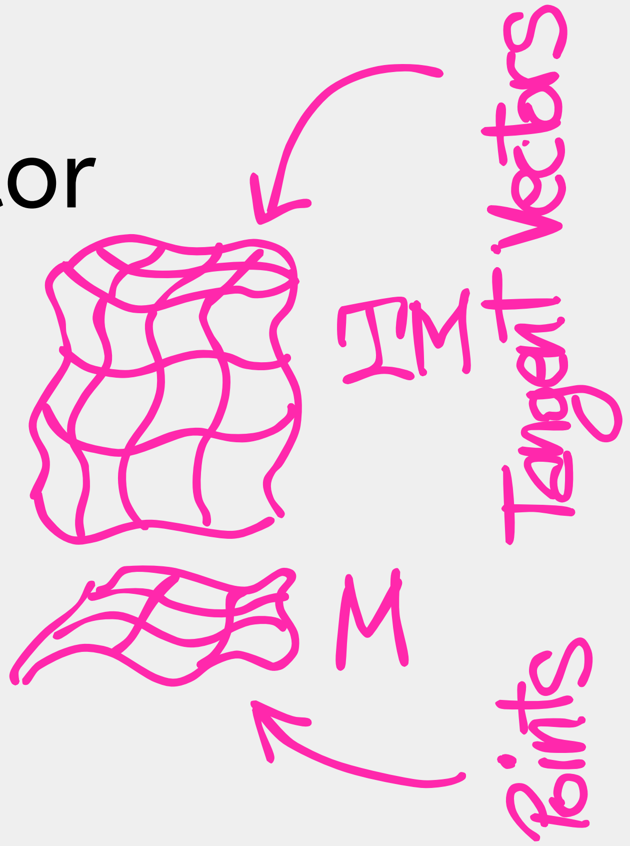


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$$p_M : \mathbb{T}M \longrightarrow M$$

Zero morphism

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DEFINITION

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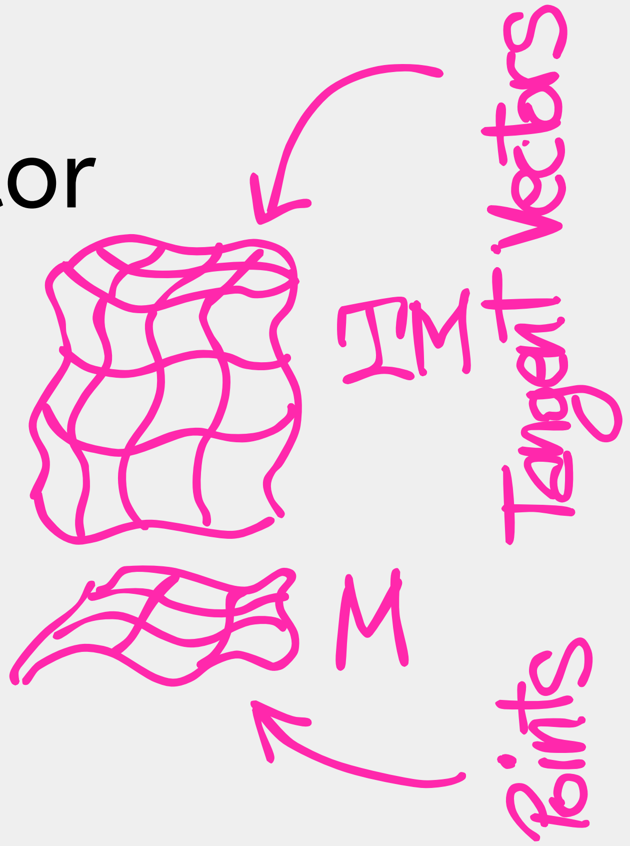


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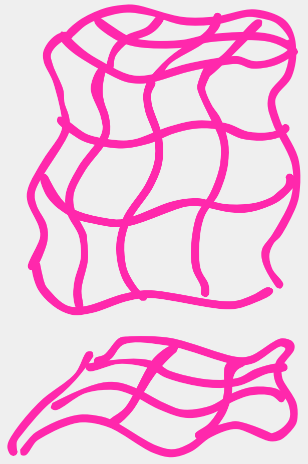
Additive structure on the tangent spaces



$\mathbb{T}M$ Tangent vectors

$M$ Points

DEFINITION

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COCKETT CRUTTWELL 2014

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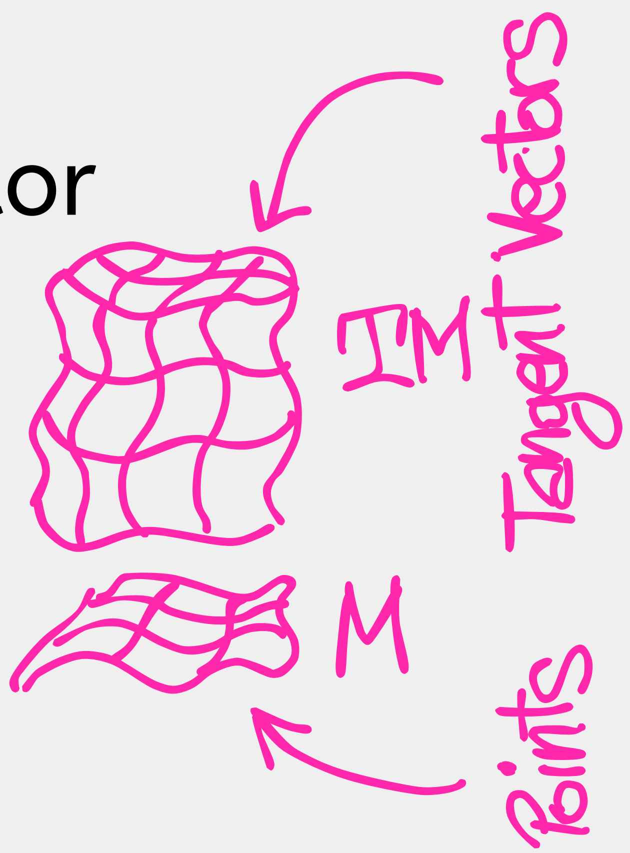


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Additive structure on the tangent spaces

Vertical lift

$$\ell_M : \mathsf{T}M \longrightarrow \mathsf{T}^2 M$$

Local Linearity:

$$\mathsf{T}\mathsf{T}_x M = \mathsf{T}_x M \times \mathsf{T}_x M$$

# DEFINITION

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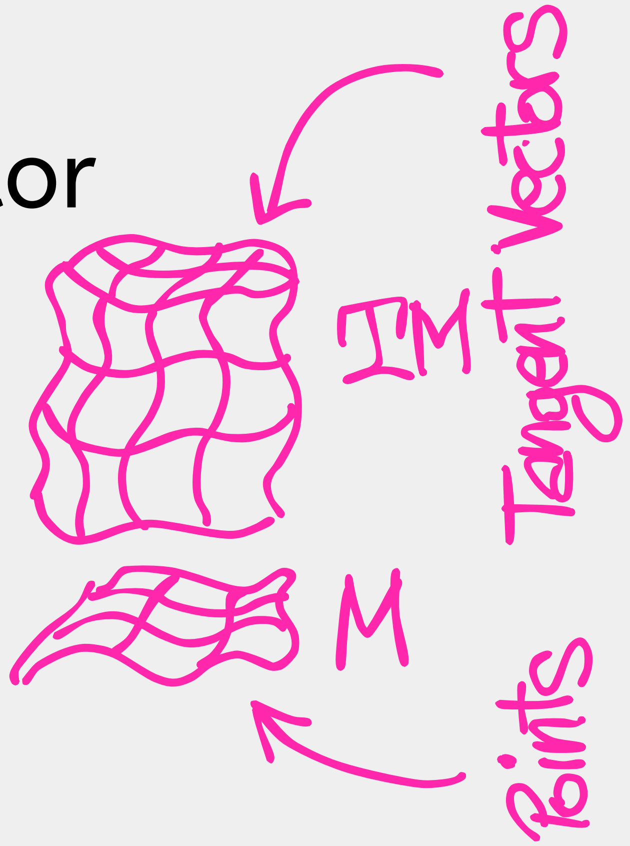


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Symmetry of partial derivatives

$\partial_x \partial_y = \partial_y \partial_x$

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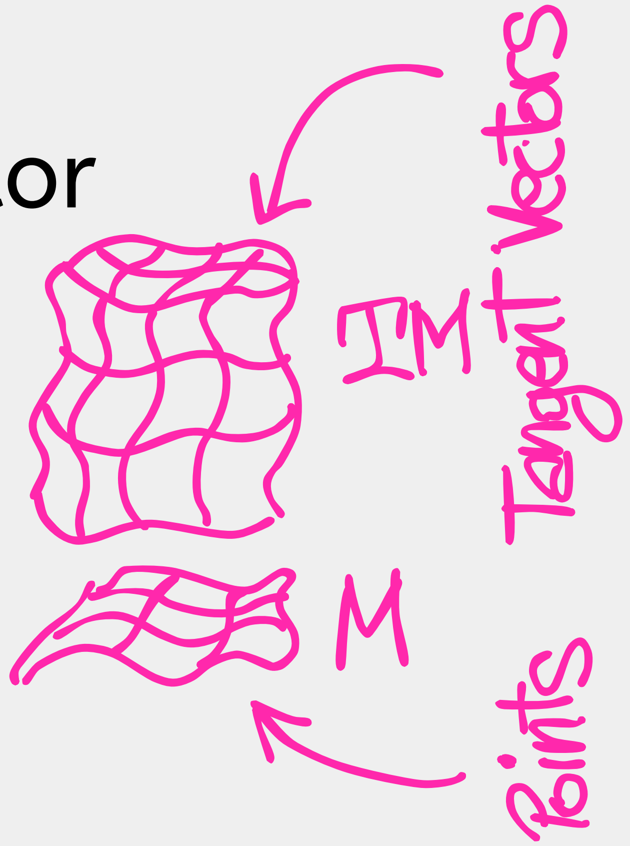


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$\partial_x \partial_y = \partial_y \partial_x$

Negation

$n_M : \mathbb{T}M \to \mathbb{T}M$

Abelian group structure on the fibres

DEFINITION

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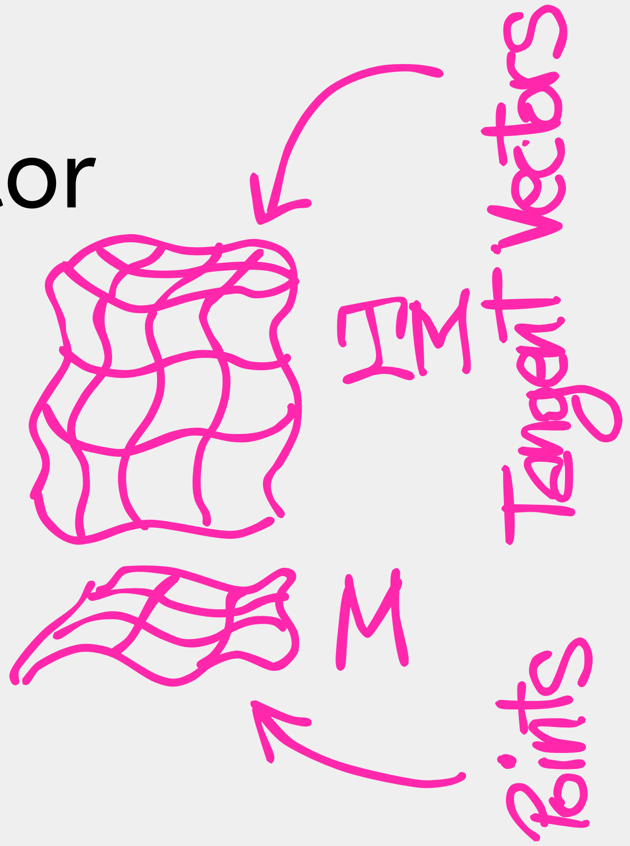


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Abelian group structure on the fibres

+ Axioms

DEFINITION

# VECTOR FIELDS

ROSICKÝ 1984  COCKETT CRUTTWELL 2014

A **vector field** in a tangent category consists of a morphism

$$v : M \longrightarrow TM$$

which is a section of the projection:

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[dr, equal] & TM \arrow[d, "p"] \\ & M \end{tikzcd}$$

# VECTOR FIELDS

ROSICKÝ 1984 COCKETT CRUTTWELL 2014

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Vector fields form a tangent category $VF(\mathbb{X}, \mathbb{T})$

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**Objects:** $(M, v)$

DEFINITION

# VECTOR FIELDS

ROSICKÝ 1984 — COCKETT CRUTTWELL 2014

A **vector field** in a tangent category consists of a morphism

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Vector fields form a tangent category $\mathsf{VF}(\mathbb{X}, \mathbb{T})$

**Objects:** $(M, v)$

**Morphisms:**

$$f : (M, v) \longrightarrow (N, u)$$

$$\begin{tikzcd} \mathbb{T}M \arrow[r, "\mathbb{T}f"] & \mathbb{T}N \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, "u"'] \end{tikzcd}$$

# VECTOR FIELDS

ROSICKÝ 1984 COCKETT CRUTTWELL 2014

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Vector fields form a tangent category $\mathsf{VF}(\mathbb{X}, \mathbb{T})$

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$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, "u"'] \end{tikzcd}$$

**Tangent bundle functor:**

$$T^{VF}(M, v) := (TM, v_T)$$

$$v_T : TM \xrightarrow{Tv} T^2M \xrightarrow{c} T^2M$$

DEFINITION

# VECTOR FIELDS

ROSICKÝ 1984

COCKETT CRUTTWELL 2014

A **vector field** in a tangent category consists of a morphism

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**Tangent bundle functor:**

$$\mathbb{T}^{\mathsf{VF}}(M, v) := (\mathbb{T}M, v_{\mathbb{T}})$$

$$v_{\mathbb{T}} : \mathbb{T}M \xrightarrow{\mathbb{T}v} \mathbb{T}^2M \xrightarrow{c} \mathbb{T}^2M$$

**Natural transformations:**
As in the base tangent category.

CHAPTER 2

# TANGENTADS

# TANGENTADS

LANFRANCHI 2024

DEFINITION

Object in a 2-category $\mathbb{X}$

DEFINITION

# TANGENTADS

LANFRANCHI 2024

Object in a 2-category $\mathbb{X}$

Tangent bundle 1-morphism

$\mathrm{T}: \mathbb{X} \to \mathbb{X}$

DEFINITION

# TANGENTADS

LANFRANCHI 2024

Object in a 2-category $\mathbb{X}$

Tangent bundle 1-morphism

$T : \mathbb{X} \to \mathbb{X}$

Projection

$p : T \Rightarrow \mathrm{id}_{\mathbb{X}}$

# TANGENTADS

LANFRANCHI 2024

DEFINITION

Object in a 2-category $\mathbb{X}$

Tangent bundle 1-morphism

$\mathrm{T}:\mathbb{X}\to\mathbb{X}$

Projection

$p:\mathrm{T}\to\mathrm{id}_{\mathbb{X}}$

Zero morphism

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DEFINITION

# TANGENTADS

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$T : \mathbb{X} \to \mathbb{X}$

Projection

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Zero morphism

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DEFINITION

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LANFRANCHI 2024

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Vertical lift

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DEFINITION

# TANGENTADS

LANFRANCHI 2024

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LANFRANCHI 2024

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Canonical flip

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Negation

$$n : T \to T$$

DEFINITION

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Negation

$n : T \to T$

+ Axioms

LEMMA

# THE HOM-TANGENT CATEGORIES



For every pairs of tangentads

$$(\mathbb{X},\mathbb{T}) \qquad (\mathbb{X}',\mathbb{T}')$$

of a 2-category **K**, the hom-category:

$$[\mathbb{X}',\mathbb{T}';\mathbb{X},\mathbb{T}]$$

in the 2-category of tangentads, comes equipped with a tangent structure.

# THE HOM-TANGENT CATEGORIES

LANFRANCHI 2024

**LEMMA**

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**Objects:** $(F,\alpha)\colon(\mathbb{X}',\mathbb{T}')\to(\mathbb{X},\mathbb{T})$

$F\colon\mathbb{X}'\to\mathbb{X} \quad \alpha\colon F\circ\mathbb{T}'\Rightarrow\mathbb{T}\circ F$

LEMMA

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LANFRANCHI 2024

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**Objects:** $(F,\alpha)\colon(\mathbb{X}',\mathbb{T}')\to(\mathbb{X},\mathbb{T})$

$F\colon\mathbb{X}'\to\mathbb{X} \quad \alpha\colon F\circ\mathrm{T}'\Rightarrow\mathrm{T}\circ F$

**Morphisms:**

$\varphi\colon(F,\alpha)\to(G,\beta) \quad \varphi\colon F\Rightarrow G$

# THE HOM-TANGENT CATEGORIES

LANFRANCHI 2024

**LEMMA**

For every pairs of tangentads

$$(\mathbb{X},\mathbb{T}) \quad (\mathbb{X}',\mathbb{T}')$$

of a 2-category **K**, the hom-category:

$$[\mathbb{X}',\mathbb{T}';\mathbb{X},\mathbb{T}]$$

in the 2-category of tangentads, comes equipped with a tangent structure.

**Objects:** $(F,\alpha):(\mathbb{X}',\mathbb{T}')\to(\mathbb{X},\mathbb{T})$

$F:\mathbb{X}'\to\mathbb{X} \quad \alpha:F\circ\mathrm{T}'\Rightarrow\mathrm{T}\circ F$

**Morphisms:**

$\varphi:(F,\alpha)\to(G,\beta) \quad \varphi:F\Rightarrow G$

**Tangent bundle functor:**

$$\mathbb{T}(F,\alpha):(\mathbb{X}',\mathbb{T}')\xrightarrow{(F,\alpha)}(\mathbb{X},\mathbb{T})\xrightarrow{(\mathrm{T},c)}(\mathbb{X},\mathbb{T})$$

# THE HOM-TANGENT CATEGORIES

## LEMMA

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$\varphi:(F,\alpha)\to(G,\beta) \quad \varphi:F\Rightarrow G$

**Tangent bundle functor:**

$$\mathbb{T}(F,\alpha):(\mathbb{X}',\mathbb{T}')\xrightarrow{(F,\alpha)}(\mathbb{X},\mathbb{T})\xrightarrow{(\mathrm{T},c)}(\mathbb{X},\mathbb{T})$$

**Natural transformations:**
As in the base tangentad.

CHAPTER 3

# THE UNIVERSAL PROPERTY OF VECTOR FIELDS

SET UP

# UNIVERSAL PROPERTY OF VECTOR FIELDS

Consider a tangent category

$$(\mathbb{X}, \mathbb{T})$$

and the forgetful functor

$$U : VF(\mathbb{X}, \mathbb{T}) \to (\mathbb{X}, \mathbb{T})$$

$$\begin{tikzcd} (M, v) \arrow[d, "f"'] & \mapsto & M \arrow[d, "f"] \\ (N, u) & & N \end{tikzcd}$$

SET UP

# UNIVERSAL PROPERTY OF VECTOR FIELDS

Consider a tangent category

$$(\mathbb{X},\mathbb{T})$$

and the forgetful functor

$$U : \mathrm{VF}(\mathbb{X},\mathbb{T}) \longrightarrow (\mathbb{X},\mathbb{T})$$

$$\begin{array}{ccc} (M,v) & & M \\ f\downarrow & \mapsto & \downarrow f \\ (N,u) & & N \end{array}$$

$$U \in [\mathrm{VF}(\mathbb{X},\mathbb{T}); \mathbb{X},\mathbb{T}]$$

# UNIVERSAL PROPERTY OF VECTOR FIELDS

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$$U \in [VF(\mathbb{X}, \mathbb{T}); \mathbb{X}, \mathbb{T}]$$

$$\frac{\hat{v} : U \longrightarrow \mathbb{T}U}{\dfrac{U(M, v) \dashrightarrow \mathbb{T}U(M, v)}{M \dashrightarrow \mathbb{T}M}}$$

# UNIVERSAL PROPERTY OF VECTOR FIELDS

**SET UP**

Consider a tangent category

$$(\mathbb{X},\mathbb{T})$$

and the forgetful functor

$$U : \mathsf{VF}(\mathbb{X},\mathbb{T}) \longrightarrow (\mathbb{X},\mathbb{T})$$

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$$U \in [\mathsf{VF}(\mathbb{X},\mathbb{T}); \mathbb{X},\mathbb{T}]$$

$$\hat{v} : U \longrightarrow \mathbb{T}U$$

$$U(M,v) \dashrightarrow \mathbb{T}U(M,v)$$

$$M \overset{v}{\dashrightarrow} \mathbb{T}M$$

# UNIVERSAL PROPERTY OF VECTOR FIELDS

**SET UP**

Consider a tangent category

$$(\mathbb{X}, \mathbb{T})$$

and the forgetful functor

$$\begin{array}{rcl} \mathcal{U} : \mathrm{VF}(\mathbb{X}, \mathbb{T}) & \longrightarrow & (\mathbb{X}, \mathbb{T}) \\ (M, v) & & M \\ f \big\downarrow & \mapsto & \big\downarrow f \\ (N, u) & & N \end{array}$$

$$\mathcal{U} \in [\mathrm{VF}(\mathbb{X}, \mathbb{T}); \mathbb{X}, \mathbb{T}]$$

$$\hat{v} : \mathcal{U} \longrightarrow \mathbb{T}\mathcal{U}$$

$$\mathcal{U}(M, v) \dashrightarrow \mathbb{T}\mathcal{U}(M, v)$$

$$M \overset{v}{\dashrightarrow} \mathbb{T}M$$

$*$ $\hat{v}$ natural $\Leftrightarrow$

$$\begin{tikzcd} \mathbb{T}M \arrow[r, "\mathbb{T}f"] & \mathbb{T}N \\ M \arrow[u, "v"] \arrow[r, "f"] & N \arrow[u, "u"'] \end{tikzcd}$$

# UNIVERSAL PROPERTY OF VECTOR FIELDS

## SET UP

Consider a tangent category

$$(\mathbb{X},\mathbb{T})$$

and the forgetful functor

$$U : VF(\mathbb{X},\mathbb{T}) \longrightarrow (\mathbb{X},\mathbb{T})$$

$$\begin{tikzcd}
(M,v) \arrow[d, "f"'] & \mapsto & M \arrow[d, "f"] \\
(N,u) & & N
\end{tikzcd}$$

$$U \in [VF(\mathbb{X},\mathbb{T}); \mathbb{X},\mathbb{T}]$$

$$\hat{v} : U \longrightarrow \mathbb{T}U$$

$$U(M,v) \dashrightarrow \mathbb{T}U(M,v)$$

$$M \overset{v}{\dashrightarrow} \mathbb{T}M$$

* $\hat{v}$ natural $\Leftrightarrow$

$$\begin{tikzcd}
\mathbb{T}M \arrow[r, "\mathbb{T}f"] & \mathbb{T}N \\
M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, "u"']
\end{tikzcd}$$

* $\hat{v}$ tangent $\Leftrightarrow$

$$\mathbb{T}^{VF}(M,v) = (\mathbb{T}M, \mathbb{T}M \xrightarrow{\mathbb{T}v} \mathbb{T}^2M \xrightarrow{c} \mathbb{T}^2M)$$

SET UP

# UNIVERSAL PROPERTY OF VECTOR FIELDS

Consider a tangent category

$$(\mathbb{X},\mathbb{T})$$

and the forgetful functor

$$\begin{tikzcd} U : VF(\mathbb{X},\mathbb{T}) \arrow[r] & (\mathbb{X},\mathbb{T}) \end{tikzcd}$$

$$\begin{tikzcd} (M,v) \arrow[d, "f"'] & & M \arrow[d, "f"] \\ (N,u) & \mapsto & N \end{tikzcd}$$

$$U \in [VF(\mathbb{X},\mathbb{T}); \mathbb{X},\mathbb{T}]$$

$$\frac{\hat{v} : U \longrightarrow \mathbb{T}U}{\dfrac{U(M,v) \dashrightarrow \mathbb{T}U(M,v)}{M \overset{v}{\dashrightarrow} \mathbb{T}M}}$$

* $\hat{v}$ natural $\Leftrightarrow$

$$\begin{tikzcd} \mathbb{T}M \arrow[r, "\mathbb{T}f"] & \mathbb{T}N \\ M \arrow[u, "v"] \arrow[r, "f"] & N \arrow[u, "u"'] \end{tikzcd}$$

* $\hat{v}$ tangent $\Leftrightarrow$

$$\mathbb{T}^{VF}(M,v) = (\mathbb{T}M, \mathbb{T}M \xrightarrow{\mathbb{T}v} \mathbb{T}^2M \xrightarrow{c} \mathbb{T}^2M)$$

* $\hat{v}$ vector field in $[VF(\mathbb{X},\mathbb{T}); \mathbb{X},\mathbb{T}]$

THEOREM

# UNIVERSAL PROPERTY OF VECTOR FIELDS

LANFRANCHI 2025

The vector field

$$\hat{v} : \mathcal{U} \longrightarrow \mathrm{T}\mathcal{U}$$

in the Hom-tangent category

$$[\mathsf{VF}(\mathbb{X}, \mathbb{T}); \mathbb{X}, \mathbb{T}]$$

is the **universal vector field**

THEOREM

# UNIVERSAL PROPERTY OF VECTOR FIELDS

LANFRANCHI 2025

The vector field

$$\hat{v} : \mathcal{U} \longrightarrow \mathbb{T}\mathcal{U}$$

in the Hom-tangent category

$$[\mathrm{VF}(\mathbb{X},\mathbb{T}'); \mathbb{X},\mathbb{T}']$$

is the **universal vector field**,
namely, the induced functors

$$[\mathbb{X}',\mathbb{T}'; \mathrm{VF}(\mathbb{X},\mathbb{T}')] \to \mathrm{VF}[\mathbb{X}',\mathbb{T}'; \mathbb{X},\mathbb{T}]$$

$$\left((\mathbb{X}',\mathbb{T}') \xrightarrow{(F,\alpha)} \mathrm{VF}(\mathbb{X},\mathbb{T})\right) \mapsto (\mathbb{X}',\mathbb{T}') \xrightarrow{(F,\alpha)} \mathrm{VF}(\mathbb{X},\mathbb{T}) \begin{tikzcd} {} \arrow[r, bend left, "\mathbb{T}\mathcal{U}"] \arrow[r, bend right, "\mathcal{U}"'] \arrow[r, phantom, "\Uparrow \hat{v}"] & {} \end{tikzcd} (\mathbb{X},\mathbb{T}')$$

are isomorphisms of tangent
categories.

# FORMAL VECTOR FIELDS



DEFINITION

A tangentad

$$(\mathbb{X},\mathbb{T})$$

**admits the construction of vector fields** if
there exists a tangentad

$$\mathrm{VF}(\mathbb{X},\mathbb{T})$$

together with a vector field

$$\hat{v}: U \to \mathbb{T}U$$

of the Hom-tangent category

$$[\mathrm{VF}(\mathbb{X},\mathbb{T});\mathbb{X},\mathbb{T}]$$

whose induced functors

$$[\mathbb{X}',\mathbb{T}';\mathrm{VF}(\mathbb{X},\mathbb{T})] \to \mathrm{VF}[\mathbb{X}',\mathbb{T}';\mathbb{X},\mathbb{T}]$$

are isomorphisms of tangent categories.

CHAPTER 4

# SOME CONSTRUCTIONS

# ADDITIVITY & LIE BRACKET



**THEOREM**

$$(U : VF(\mathbb{X},\mathbb{T}) \to (\mathbb{X},\mathbb{T});\, 0,\, +,\, [,])$$

forms a Lie algebra in the slice category

$$\mathsf{TNG}(\mathbb{K})/(\mathbb{X},\mathbb{T})$$

THE END.

# THANKS

ARXIV
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