The magnitude of a presheaf

Tom Leinster

University of Edinburgh and the Maxwell Institute

These slides: on my web page

Background

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Magnitude homology		

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Magnitude homology	Homology of $B\mathbf{A}$	Detects non-uniqueness of geodesics, size of holes, ...

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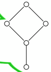

Möbius inversion for posets

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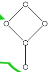

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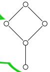

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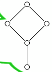

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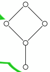

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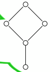

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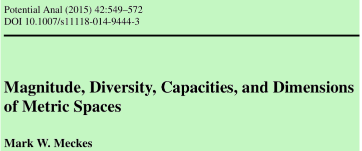
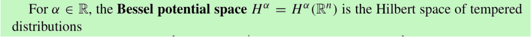


Potential Anal (2015) 42:549–572
DOI 10.1007/s11118-014-9444-3

Magnitude, Diversity, Capacities, and Dimensions of Metric Spaces

Mark W. Meckes



For $\alpha \in \mathbb{R}$, the **Bessel potential space** $H^\alpha = H^\alpha(\mathbb{R}^n)$ is the Hilbert space of tempered distributions

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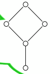

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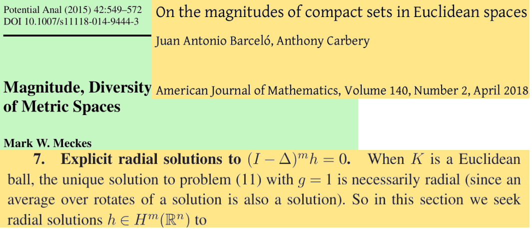

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Magnitude, Diversity of Metric Spaces

Mark W. Meckes

On the magnitudes of compact sets in Euclidean spaces

Juan Antonio Barceló, Anthony Carbery

American Journal of Mathematics, Volume 140, Number 2, April 2018

7. Explicit radial solutions to $(I-\Delta)^m h = 0$. When K is a Euclidean ball, the unique solution to problem (11) with $g = 1$ is necessarily radial (since an average over rotates of a solution is also a solution). So in this section we seek radial solutions $h \in H^m(\mathbb{R}^n)$ to

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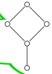

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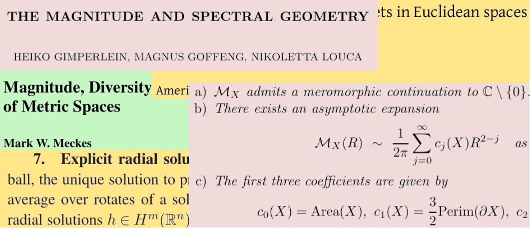

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HEIKO GIMPERLEIN, MAGNUS GOFFENG, NIKOLETTA LOUCA

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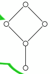

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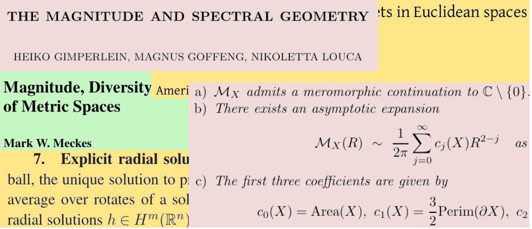

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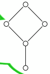

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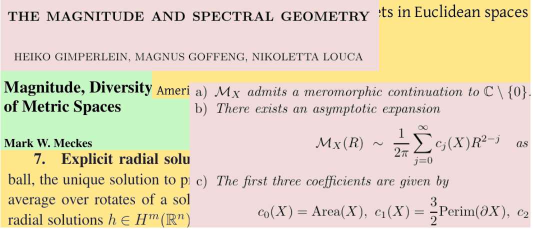

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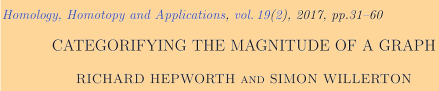
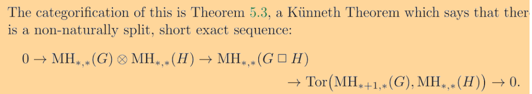


Homology, Homotopy and Applications, vol. 19(2), 2017, pp.31–60

CATEGORIFYING THE MAGNITUDE OF A GRAPH

RICHARD HEPWORTH AND SIMON WILLERTON



The categorification of this is Theorem 5.3, a Künneth Theorem which says that ther is a non-naturally split, short exact sequence:

$$0 \to \mathrm{MH}_{*,*}(G) \otimes \mathrm{MH}_{*,*}(H) \to \mathrm{MH}_{*,*}(G \,\square\, H)$$
$$\to \mathrm{Tor}(\mathrm{MH}_{*+1,*}(G), \mathrm{MH}_{*,*}(H)) \to 0.$$

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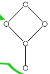

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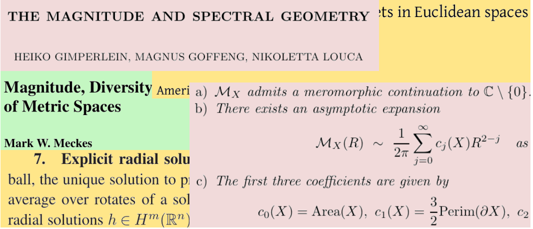

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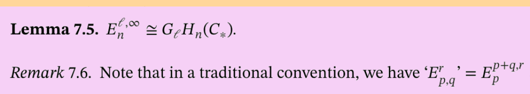
RESEARCH ARTICLE

Bulletin of the London
Mathematical Society

Magnitude homology and path homology

Yasuhiko Asao



Lemma 7.5. $E_n^{\ell,\infty} \cong G_\ell H_n(C_*)$.

Remark 7.6. Note that in a traditional convention, we have ‘$E^r_{p,q}$’ $= E_p^{p+q,r}$

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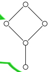

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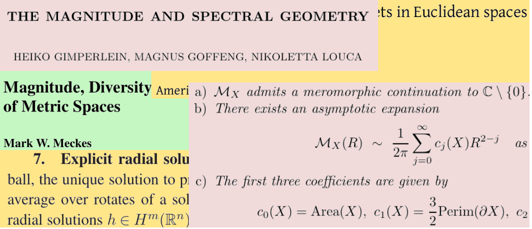

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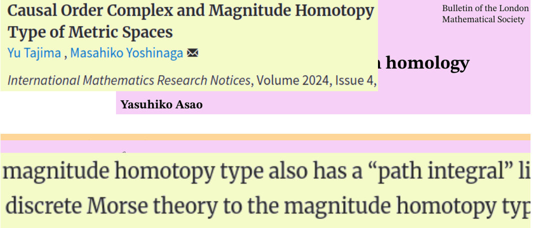

Causal Order Complex and Magnitude Homotopy Type of Metric Spaces

Yu Tajima , Masahiko Yoshinaga ✉

International Mathematics Research Notices, Volume 2024, Issue 4,

Bulletin of the London Mathematical Society

homology

Yasuhiko Asao

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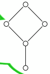

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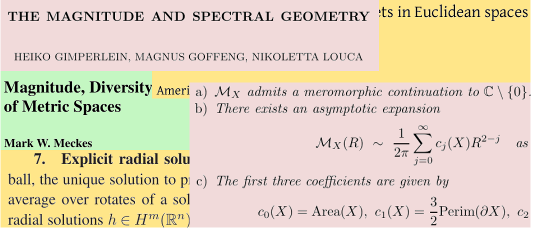

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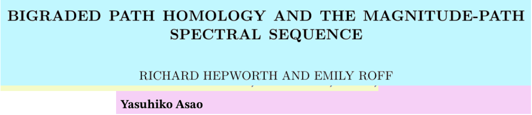
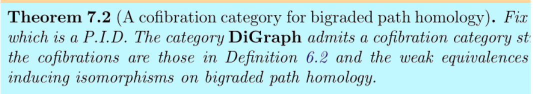


BIGRADED PATH HOMOLOGY AND THE MAGNITUDE-PATH SPECTRAL SEQUENCE

RICHARD HEPWORTH AND EMILY ROFF

Yasuhiko Asao



Theorem 7.2 (A cofibration category for bigraded path homology). *Fix which is a P.I.D. The category* **DiGraph** *admits a cofibration category st the cofibrations are those in Definition 6.2 and the weak equivalences inducing isomorphisms on bigraded path homology.*

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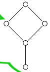

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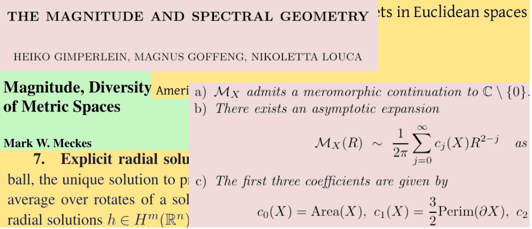

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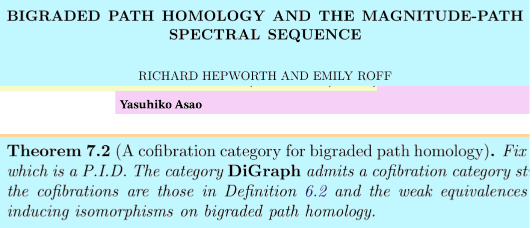

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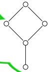

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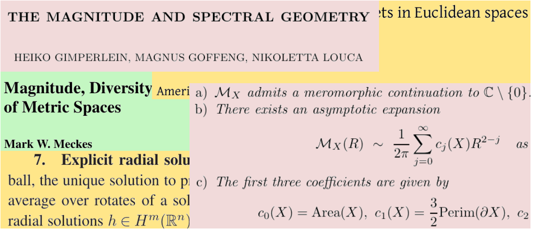

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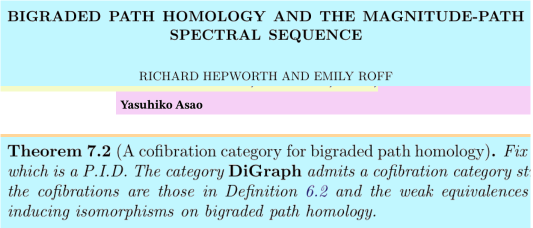

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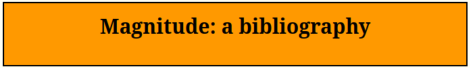

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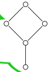

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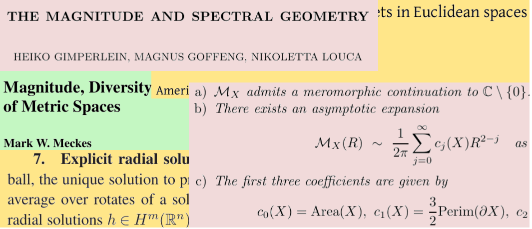

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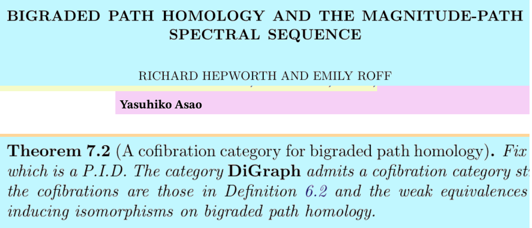

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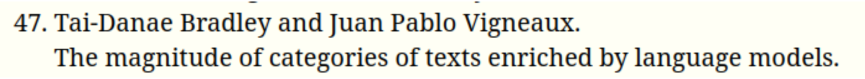

47. Tai-Danae Bradley and Juan Pablo Vigneaux.
The magnitude of categories of texts enriched by language models.

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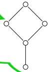

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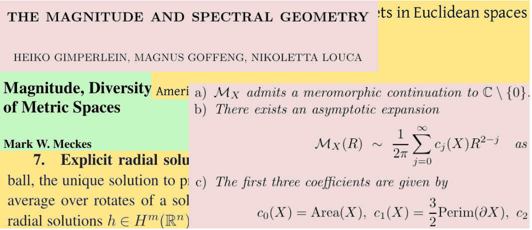

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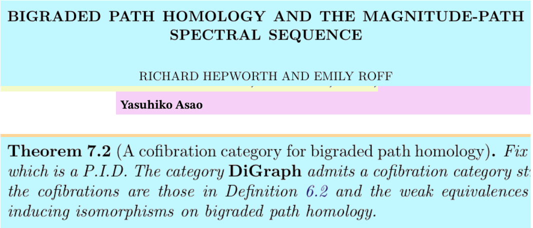

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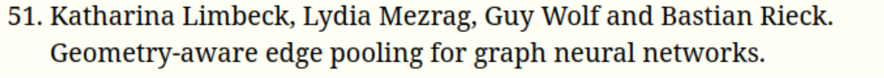

51. Katharina Limbeck, Lydia Mezrag, Guy Wolf and Bastian Rieck. Geometry-aware edge pooling for graph neural networks.

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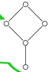

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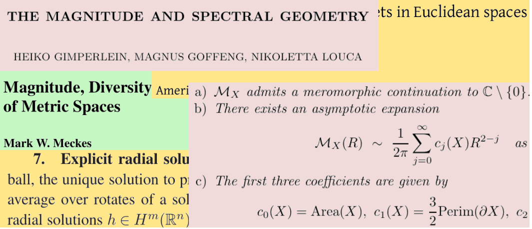

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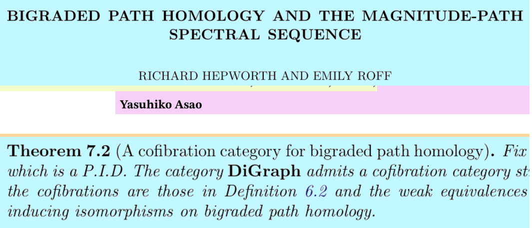

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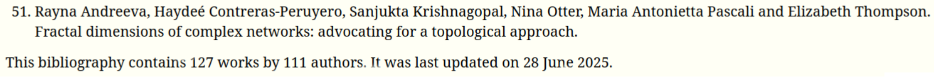

51. Rayna Andreeva, Haydeé Contreras-Peruyero, Sanjukta Krishnagopal, Nina Otter, Maria Antonietta Pascali and Elizabeth Thompson. Fractal dimensions of complex networks: advocating for a topological approach.

This bibliography contains 127 works by 111 authors. It was last updated on 28 June 2025.

From categories to presheaves

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Analogy:

magnitude of categories	magnitude of presheaves
measure	integration

Plan

1. Definitions
2. Examples
3. Comagnitude

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Under mild hypotheses, $|A|$ equals $\chi(B\mathbf{A})$, the Euler characteristic of the classifying space.

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- Free action of monoid M on set X: number of orbits is $|X|\,/\mathrm{order}(M)$.

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  of presheaves,

$$|X_4| = |X_2| + |X_3| - |X_1| \, .$$

Enriching the definitions

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The definition extends to many *compact* metric spaces.

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- For an action of a monoid M on a set X, the magnitude of the corresponding functor $M \to \mathbf{Set}$ is

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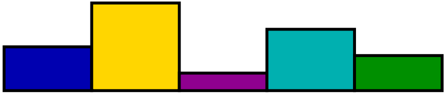

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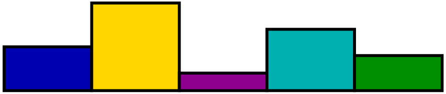

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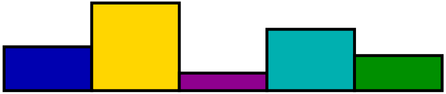

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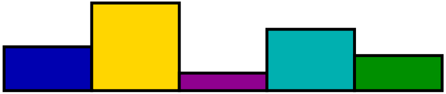

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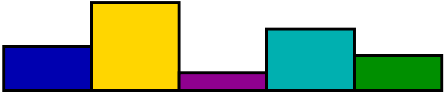

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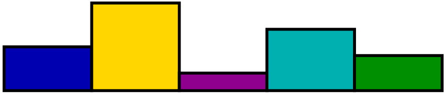

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Theorem *The magnitude of this functor is* $e^{H(\mu)}$, *the exponential of the entropy of the measure.*

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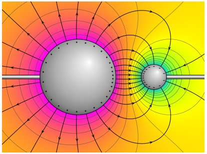

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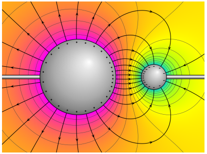

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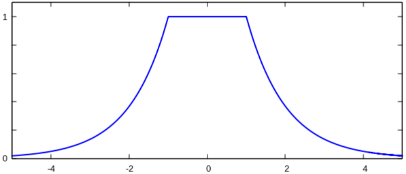

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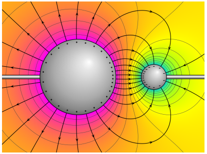

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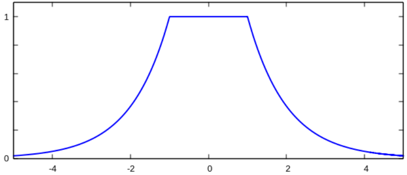

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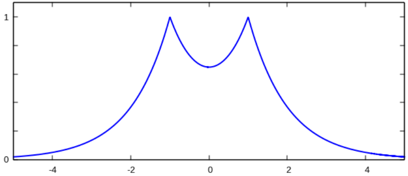

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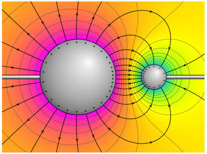

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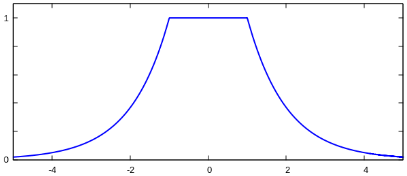

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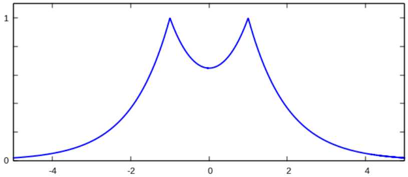

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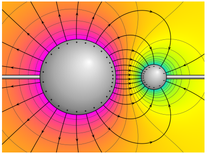

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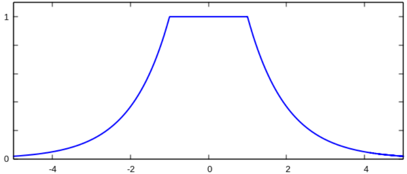

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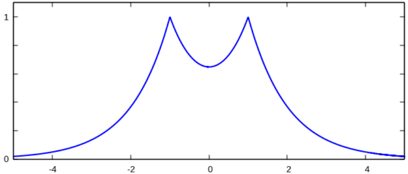

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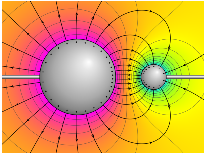

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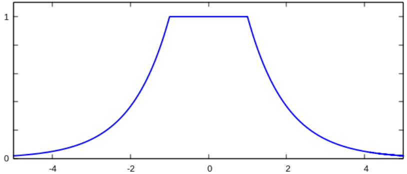

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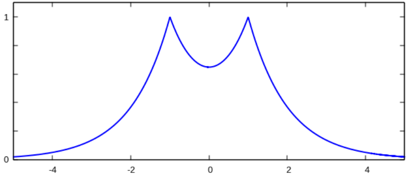

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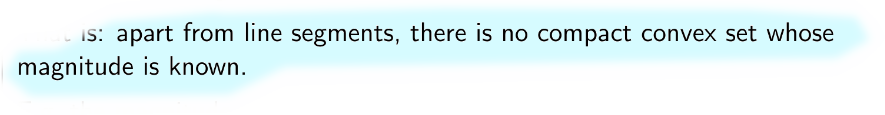

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How potential functions came to the rescue

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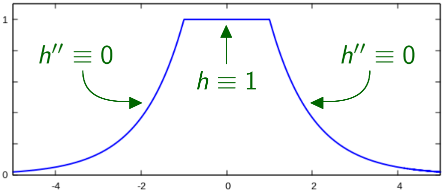

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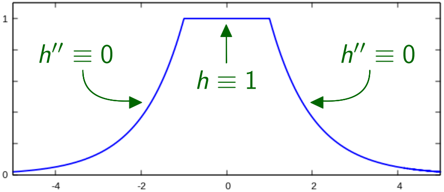

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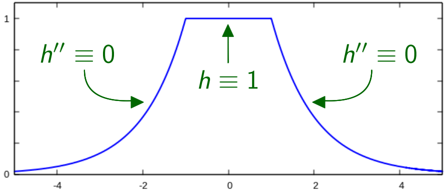

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[Möbius] inversion can be carried out by the analog of the "difference operator" relative to a partial ordering

Gian-Carlo Rota, 1964

3. Comagnitude

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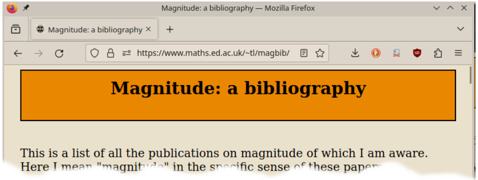
Theorem *The result is the comagnitude of X, assuming that $\mathbf{A}$ is the free category on a graph.*

References



Magnitude: a bibliography

This is a list of all the publications on magnitude of which I am aware. Here I mean "magnitude" in the specific sense of these papers

www.maths.ed.ac.uk/∼tl/magbib

Blog posts on the *n*-Category Café (2025):

- Comagnitude 1, 2
- Potential functions and the magnitude of functors 1, 2

Thanks for listening

Conceivably asked questions

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The magnitude homology paper by Leinster and Shulman (arXiv version 2, Section 3) has something in this direction.

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For an integer-valued joint measure as a diagram of sets $I \leftarrow S \to J$, the codiscrepancy of the presheaf

$$\begin{tikzcd}
 & \mathrm{End}_I(A) \arrow[d, tail] \\
\mathrm{End}_J(A) \arrow[r, tail] & \mathrm{End}(A)
\end{tikzcd}$$

is the exponential of the mutual information.

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The special case $\mathbf{A} = \mathbf{1}$ or $\mathbf{B} = \mathbf{1}$ reduces to the definition of the magnitude of a presheaf.