# Differential Graded Algebras in Differential Categories

**JS PL**[2] (he/him)
based on joint work with Chiara Sava
but a story involving many people!

Email: js.lemay@mq.edu.au
Website: https://sites.google.com/view/jspl-personal-webpage

**CT2025**

Thank you to the Scientific Committee and Local Organisers
and thank you to you all for still being here on the last day!

[2] Je dédie ma presentation à ma grand-maman France (23 fev 1936 – 3 juin 2025). Je t'aime. Merci pour le t-shirt.

# Dedication to Phil Scott (1947 – 2023)

- Phil was one of the giants in our (Canadian) category theory community. Phil made fundamental contributions to in category theory, (linear) logic, and theoretical computer science, having worked on topics such as categorical logic, traced monoidal categories, MV algebras, etc.
- Of course Phil is probably best known for his collaborations with J. Lambek on categorical logic and categorical proof theory, and, in particular, their all-important landmark book "Introduction To Higher-Order Categorical Logic"
- Phil was also such a kind and friendly man, an excellent mentor, and very supportive
- Phil was one of my favourite teachers from undergrad, who later became my friend, colleague, and mentor. I credit Phil with the direction of my career: as he was the one who first introduced me to category theory and put me on this career path.

# What is the Theory of Differential Categories About?

# What is the Theory of Differential Categories About?

- The theory of differential categories uses category theory to provide the foundations of differentiation and has been able to formalize numerous aspects of differential calculus.

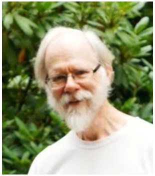
# What is the Theory of Differential Categories About?

- The theory of differential categories uses category theory to provide the foundations of differentiation and has been able to formalize numerous aspects of differential calculus.

- Originally, Blute, Cockett, and Seely

*R.Blute*

*R.Cockett*



*R.A.G.Seely*

introduced differential categories in:

R. Blute, R. Cockett, R.A.G. Seely, **Differential Categories**, (2006)

to provide the categorical semantics of Ehrhard and Regnier's Differential Linear Logic, differential $\lambda$-calculus, and differential proof nets.

- Differential categories are successful because they capture both the classical limit definition of differentiation and the more algebraic synthetic definition of differentiation. This has led to the categorical formalization of various aspects of differentiation, which is why differential categories have been become quite popular in both mathematics and computer science.

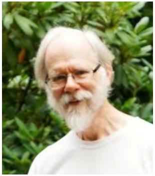
## The Differential Category World: The Four Tomes

# The Differential Category World: The Four Tomes

- **Differential Categories** (2006):
  Algebraic Foundations of Differentiation

*R.Blute*

*R.Cockett*



*R.A.G.Seely*

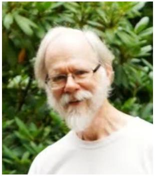
Blute, R., Cockett, R., Seely, R.A.G. **Differential Categories** (2006)

# The Differential Category World: The Four Tomes

- **Differential Categories** (2006): Algebraic Foundations of Differentiation
- **Cartesian Differential Categories** (2009):
  Foundations of Differential Calculus over Euclidean Spaces $\mathbb{R}^n$

*R.Blute*

*R.Cockett*



*R.A.G.Seely*

Blute, R., Cockett, R., Seely, R.A.G. **Cartesian Differential Categories** (2009)

# The Differential Category World: The Four Tomes

- **Differential Categories** (2006): Algebraic Foundations of Differentiation
- **Cartesian Differential Categories** (2009):
  Foundations of Differential Calculus over Euclidean Spaces $\mathbb{R}^n$
- **Differential Restriction Categories** (2011):
  Foundations of Differential Calculus over open subsets $U \subseteq \mathbb{R}^n$

*R.Cockett*

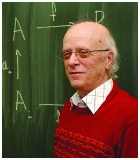
*G.Cruttwell*

*J.Gallagher*

Cockett, R., Cruttwell, G., and Gallagher, J. **Differential Restriction Categories**. (2011)

# The Differential Category World: The Four Tomes

- **Differential Categories** (2006): Algebraic Foundations of Differentiation
- **Cartesian Differential Categories** (2009):
  Foundations of Differential Calculus over Euclidean Spaces $\mathbb{R}^n$
- **Differential Restriction Categories** (2011):
  Foundations of Differential Calculus over open subsets $U \subseteq \mathbb{R}^n$
- **Tangent Categories** (1984 & 2014):
  Foundations of Differential Calculus over Smooth Manifolds



*J.Rosický*

*R.Cockett*

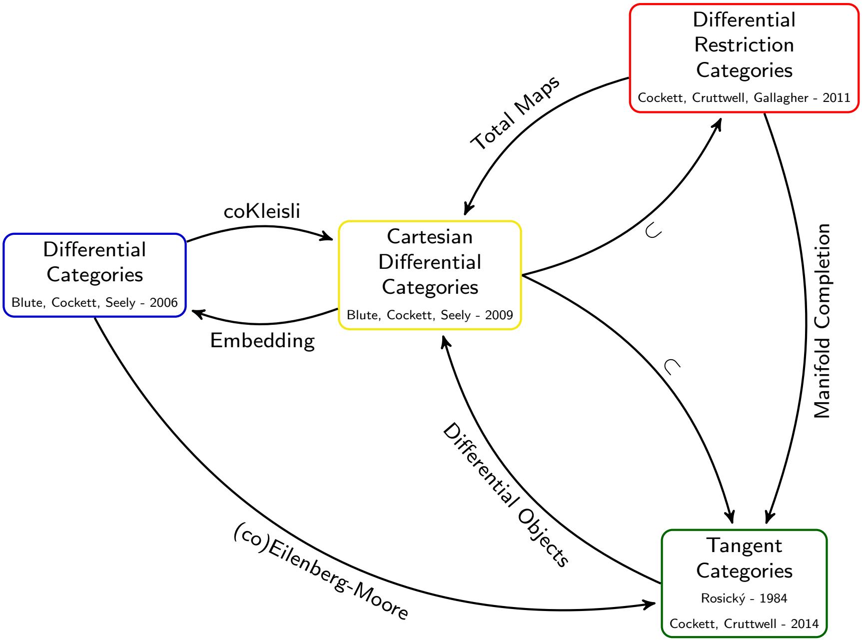
*G.Cruttwell*


J. Rosický **Abstract tangent functors** (1984) [3]

R. Cockett, G. Cruttwell **Differential structure, tangent structure, and SDG** (2014)


[3] At PSSL 106 (May 2022), which was a celebration for J. Rosický, Jiří Adámek said that Rosický's most important/influential work was now tan. cats., because he has heard a talk about tan./diff. cats. at every ct conference recently!

# The Differential Category World: It's all connected!



# The Differential Category World: The Four Tomes

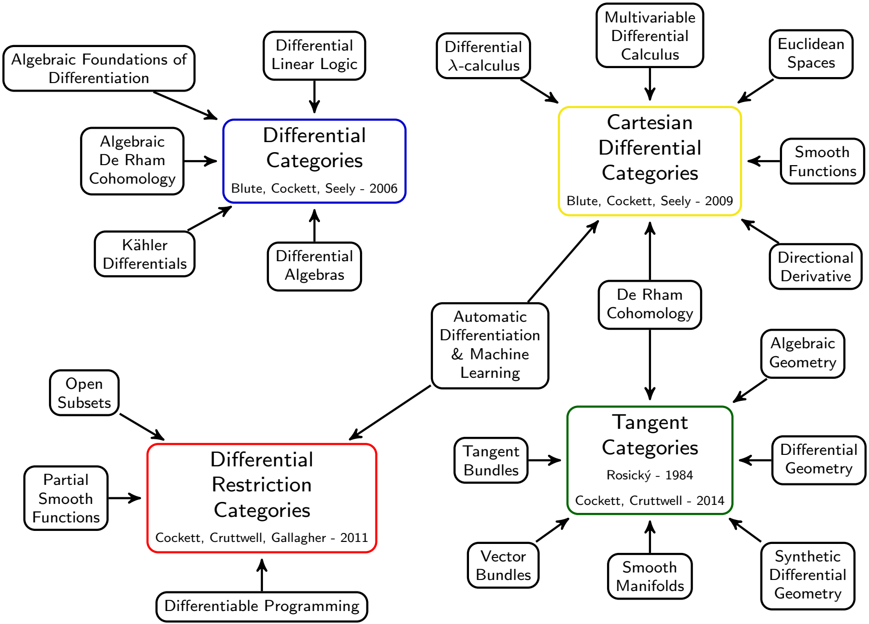
Differential Categories
Blute, Cockett, Seely - 2006

Cartesian Differential Categories
Blute, Cockett, Seely - 2009

Differential Restriction Categories
Cockett, Cruttwell, Gallagher - 2011

Tangent Categories
Rosický - 1984
Cockett, Cruttwell - 2014

# The Differential Category World: A Taster



# Snapshot of awesome people who worked on differential categories


And lots more! : like F. Schwartz, L. Regnier, N. Gambino, E. Rielh, A. Walch, etc. I just ran out of space (or I could not find a photo of them!)

Hopefully new people will work on differential categories so I can keep adding more photos!

# A frequent question I get is...

Q: Where do differential graded algebras fit into this story?

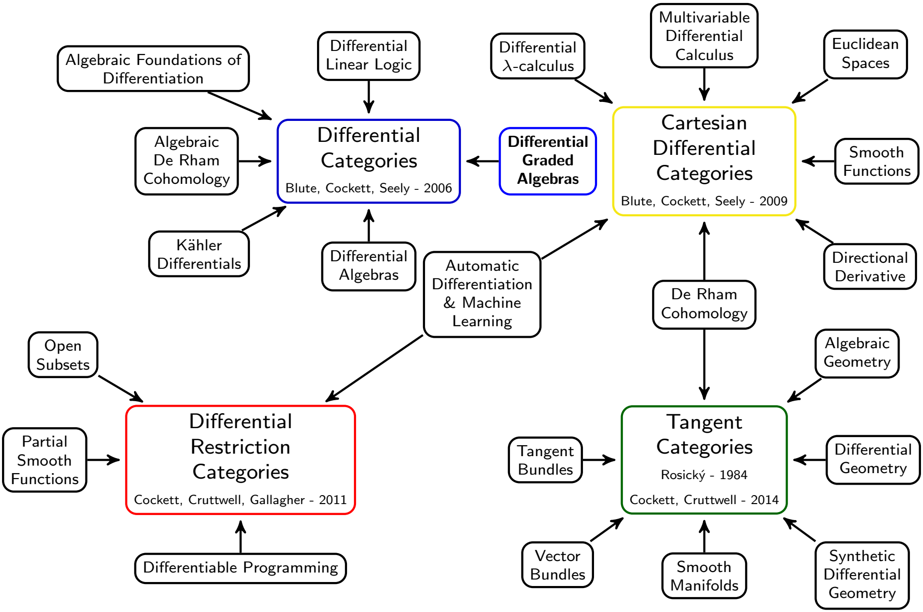
What about dg-categories?

# Enter Chiara Sava (PhD Student, Charles University)

- Chiara reach out to me saying she wanted to work on differential categories and she had all these ideas. One of them that we ended up working on was to answer where differential graded algebras and dg-categories fit into the world of differential categories.

- Chiara visited us at Macquarie from Feb-April 2025, and that photo is from her farewell supper (and yes that is Steve Lack's head behind me)

# The Differential Category World: A Taster



# The Plan

**TODAY'S STORY:** Differential Graded Algebras in Differential Categories.

The plan is:

- Introduction to Differential Categories
- Derivations in Differential Categories
- Differential Graded Algebras in Differential Categories
- Justify our definition using lifting of monads. (Technical part!)

Main references for today is:


Blute, R., Cockett, R., Seely, R.A.G. **Differential Categories** (2006)

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

Blute, R., Lucyshyn-Wright, R.B.B. and O'Neill, K. **Derivations in codifferential categories**. (2016)

O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)

# Terminology: Differential Categories vs. Codifferential Categories

Differential categories were originally introduced from the point of view of Linear Logic. So they are about:

- Comonads, comonoids, coalgebras, etc.

However, if we want to talk about differentiation in algebra, we actually need the dual notion of codifferential categories:

- Monads, monoids, algebras, etc.

So differential graded algebras fit more naturally in a codifferential category.

# Terminology: Differential Categories vs. Codifferential Categories

Differential categories were originally introduced from the point of view of Linear Logic. So they are about:

- Comonads, comonoids, coalgebras, etc.

However, if we want to talk about differentiation in algebra, we actually need the dual notion of codifferential categories:

- Monads, monoids, algebras, etc.

So differential graded algebras fit more naturally in a codifferential category.

But I don't like the term codifferential category... it scares people away!

# Terminology: Differential Categories vs. Codifferential Categories

Differential categories were originally introduced from the point of view of Linear Logic. So they are about:

- Comonads, comonoids, coalgebras, etc.

However, if we want to talk about differentiation in algebra, we actually need the dual notion of codifferential categories:

- Monads, monoids, algebras, etc.

So differential graded algebras fit more naturally in a codifferential category.

But I don't like the term codifferential category... it scares people away!

So I'm going to something blasphemous... and I've been proposing:

- To call differential categories instead coalgebraic differential categories.
- To call codifferential categories instead algebraic differential categories, or just differential categories. So I am going to do this today.

Hopefully you'll agree with this convention after we see the definition... I have Rick Blute's approval on this! (Don't ask Robin...)

# Differential Categories - Definition

An (algebraic) differential category (née codifferential category) is:

- A additive symmetric monoidal category,
- With a differential modality which is:
  - An algebra modality
  - Equipped with a deriving transformation.

# Differential Categories - Definition

An (algebraic) differential category (née codifferential category) is:

- A **additive symmetric monoidal category**,
- With a differential modality which is:
  - An algebra modality
  - Equipped with a deriving transformation.

# Additive Symmetric Monoidal Categories - Definition

An additive symmetric monoidal category is a symmetric monoidal category[4]:

$$\mathbb{X} \qquad\qquad \otimes \qquad\qquad I \qquad\qquad \sigma : A \otimes B \xrightarrow{\cong} B \otimes A$$

which is enriched over commutative monoids.

- So every homset $\mathbb{X}(A, B)$ is a $k$-module, we can add maps together $f + g$, have zero maps $0$, scalar multiply maps $r \cdot f$, and composition preserves the $k$-linear structure:

$$f \circ (g + h) \circ k = f \circ g \circ k + f \circ h \circ k \qquad\qquad f \circ 0 = 0 = 0 \circ f$$

- The monoidal product $\otimes$ also preserves the $k$-linear structure:

$$f \otimes (g + h) = f \otimes g + f \otimes h \quad (f + g) \otimes h = f \otimes h + g \otimes h \quad f \otimes 0 = 0 \quad 0 \otimes f = 0$$

We need addition to talk about the Leibniz rule and zero to talk about the constant rule (we don't necessarily need scalar multiplication, but it is useful to have). Note that this definition does not assume (bi)products or negatives (these come later).

**Example**

Let $\mathbb{K}$ be a field and and let $\text{VEC}_{\mathbb{K}}$ to be the category of all $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps between them. $\text{VEC}_{\mathbb{K}}$ is additive symmetric monoidal category with the usual monoidal and additive structure.

[4] We are going to work in the strict setting for simplicity.

# Differential Categories - Definition

An (algebraic) differential category (née codifferential category) is:

- A additive symmetric monoidal category,
- With a differential modality which is:
  - An **algebra modality**
  - Equipped with a deriving transformation.

# Algebra Modality - Definition

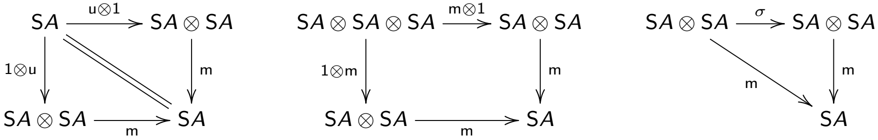
An **algebra modality** on a symmetric monoidal category is a monad

$$SSA \xrightarrow{\mu} SA \qquad\qquad A \xrightarrow{\eta} SA$$

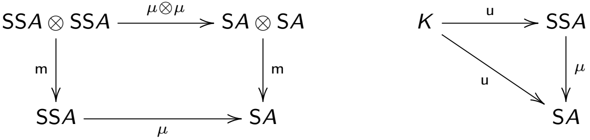
equipped with two natural transformations:

$$SA \otimes SA \xrightarrow{\mathsf{m}} SA \qquad\qquad I \xrightarrow{\mathsf{u}} SA$$

such that for every object $A$, $(SA, \mathsf{m}, \mathsf{u})$ is a commutative monoid:




and $\mu$ is a monoid morphism:



# Algebra Modality - Rough Idea

- $\mathsf{S}(A) \equiv$ set of differentiable/smooth functions $A \to I$ (whatever that means).
- $\mu \equiv$ function composition
- $\eta \equiv$ identity function/linear function
- $\mathsf{m} \equiv$ function multiplication
- $\mathsf{u} \equiv$ multiplication unit/constant function.

This idea can be made precise using (Classical) Differential Linear Logic.

# Algebra Modality - Example Polynomials

## Example

A commutative monoid in $\mathsf{VEC}_{\mathbb{K}}$ is precisely a commutative $\mathbb{K}$-algebra. Define the algebra modality $\mathsf{Sym}$ on $\mathsf{VEC}_{\mathbb{K}}$ as follows: for a $\mathbb{K}$-vector space $V$ let $\mathsf{Sym}(V)$ be the free commutative $\mathbb{K}$-algebra over $V$, also known as the free symmetric algebra on $V$.

$$\mathsf{Sym}(V) := \mathbb{K} \oplus V \oplus (V \otimes_{sym} V) \oplus \ldots = \bigoplus_{n \in \mathbb{N}} V^{\otimes^n_{sym}}$$

where $\otimes_{sym}$ is the symmetrized tensor power of $V$.

If $X = \{x_1, x_2, \ldots\}$ is a basis of $V$, then $\mathsf{Sym}(V) \cong \mathbb{K}[X]$.

In particular for $\mathbb{K}^n$, $\mathsf{Sym}(\mathbb{K}^n) \cong \mathbb{K}[x_1, \ldots, x_n]$.

# Algebra Modality - Example Polynomials

## Example

A commutative monoid in $\mathrm{VEC}_{\mathbb{K}}$ is precisely a commutative $\mathbb{K}$-algebra. Define the algebra modality Sym on $\mathrm{VEC}_{\mathbb{K}}$ as follows: for a $\mathbb{K}$-vector space $V$ let $\mathrm{Sym}(V)$ be the free commutative $\mathbb{K}$-algebra over $V$, also known as the free symmetric algebra on $V$.

$$\mathrm{Sym}(V) := \mathbb{K} \oplus V \oplus (V \otimes_{sym} V) \oplus \ldots = \bigoplus_{n \in \mathbb{N}} V^{\otimes^n_{sym}}$$

where $\otimes_{sym}$ is the symmetrized tensor power of $V$.

If $X = \{x_1, x_2, \ldots\}$ is a basis of $V$, then $\mathrm{Sym}(V) \cong \mathbb{K}[X]$.
In particular for $\mathbb{K}^n$, $\mathrm{Sym}(\mathbb{K}^n) \cong \mathbb{K}[x_1, \ldots, x_n]$.

Then the algebra modality structure can be described in terms of polynomials as (wher

$$\begin{aligned} \eta : V &\to \mathbb{K}[X] \\ x_i &\mapsto x_i \end{aligned} \qquad \begin{aligned} \mu : \mathrm{Sym}\left(\mathbb{K}[X]\right) &\to \mathbb{K}[X] \\ P\left(p_1(\vec{x_1}), \ldots, p_n(\vec{x_n})\right) &\mapsto P\left(p_1(\vec{x_1}), \ldots, p_n(\vec{x_n})\right) \end{aligned}$$

$$\begin{aligned} \mathrm{u} : \mathbb{K} &\to \mathbb{K}[X] \\ 1 &\mapsto 1 \end{aligned} \qquad \begin{aligned} \mathrm{m} : \mathbb{K}[X] \otimes \mathbb{K}[X] &\to \mathbb{K}[X] \\ p(\vec{x}) \otimes q(\vec{y}) &\mapsto p(\vec{x})q(\vec{y}) \end{aligned}$$

which we extend by linearity. Therefore, $\mu$ and $\eta$ correspond to polynomial composition, while m and u correspond to polynomial multiplication.

An (algebraic) differential category (née codifferential category) is:

- A additive symmetric monoidal category,
- With a differential modality which is:
  - An algebra modality
  - Equipped with a **deriving transformation**.

# Deriving Transformation - Definiton

A **deriving transformation** for an algebra modality on an additive symmetric monoidal category is a natural transformation:

$$SA \xrightarrow{\mathsf{d}} SA \otimes A$$

whose axioms are based on the basic identities from differential calculus.

**IDEA:** $f(x) \mapsto f'(x) \otimes \mathsf{d}x$

- **[D.1]**: Constant rule: $c' = 0$
- **[D.2]**: Product rule: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- **[D.3]**: Linear rule: $x' = 1$
- **[D.4]**: Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$
- **[D.5]**: Interchange rule: $\dfrac{d^2 f(x,y)}{dxdy} = \dfrac{d^2 f(x,y)}{dydx}$

# Deriving Transformation - Definiton

A **deriving transformation** for an algebra modality on an additive symmetric monoidal category is a natural transformation:

$$SA \xrightarrow{\mathsf{d}} SA \otimes A$$

whose axioms are based on the basic identities from differential calculus.

**IDEA:** $f(x) \mapsto f'(x) \otimes \mathsf{d}x$

- **[D.1]**: Constant rule: $c' = 0$
- **[D.2]**: Product rule: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- **[D.3]**: Linear rule: $x' = 1$
- **[D.4]**: Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$
- **[D.5]**: Interchange rule: $\frac{d^2 f(x,y)}{dxdy} = \frac{d^2 f(x,y)}{dydx}$

## Example

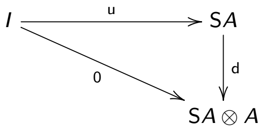
Let $V$ be a $\mathbb{K}$-vector space with basis $X = \{x_1, x_2, \ldots\}$.

The deriving transformation can be described in terms of polynomials as follows:

$$\mathsf{d} : \mathbb{K}[X] \to \mathbb{K}[X] \otimes V$$

$$\mathsf{p}(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i$$

# D.1 - Constant Rule




## Example

For a constant polynomial $\mathrm{p}(x_1, \ldots, x_n) = r$:

$$\sum_{i=1}^{n} \frac{\partial \mathrm{p}}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i = 0$$

# D.2 - Product Rule

$$\begin{array}{ccc} SA \otimes SA & \xrightarrow{(1\otimes \mathrm{d})+(1\otimes\sigma)\circ(\mathrm{d}\otimes 1)} & SA \otimes SA \otimes A \\ \Big\downarrow{\scriptstyle \mathrm{m}} & & \Big\downarrow{\scriptstyle \mathrm{m}\otimes 1} \\ SA & \xrightarrow[\mathrm{d}]{} & SA \otimes A \end{array}$$

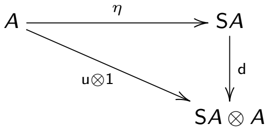
## Example

For polynomials $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$:

$$\sum_{i=1}^{n} \frac{\partial pq}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i$$

$$= \sum_{i=1}^{n} p(x_1, \ldots, x_n) \frac{\partial q}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(x_1, \ldots, x_n) q(x_1, \ldots, x_n) \otimes x_i$$

# D.3 - Linear Rule




## Example

For a monomial of degree 1, $\mathrm{p}(x_1, \ldots, x_n) = x_j$:

$$\sum_{i=1}^{n} \frac{\partial x_j}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i = 1 \otimes x_j$$

$$\begin{array}{ccccc}
SSA & & \xrightarrow{\quad\mu\quad} & & SA \\
\downarrow{\scriptstyle d} & & & & \downarrow{\scriptstyle d} \\
SSA \otimes SA & \xrightarrow[\mu \otimes d]{} & SA \otimes SA \otimes A & \xrightarrow[m \otimes 1]{} & SA \otimes A
\end{array}$$

## Example

For polynomials $p(x_1, \ldots, x_n)$ and $q(x)$:

$$\sum_{i=1}^{n} \frac{\partial q(p(x_1, \ldots, x_n))}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i = \sum_{i=1}^{n} \frac{\partial q}{\partial x}(p(x_1, \ldots, x_n)) \frac{\partial q}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i$$

# D.5 - Interchange Rule

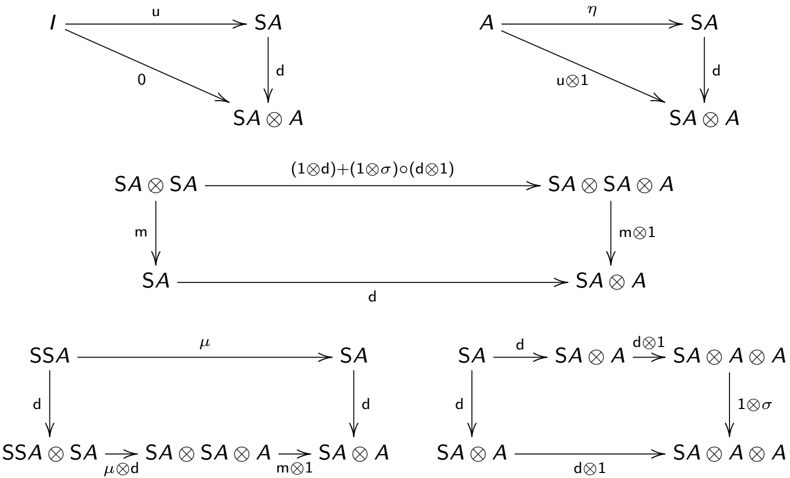
$$
\begin{array}{ccccc}
SA & \xrightarrow{\;\mathsf{d}\;} & SA \otimes A & \xrightarrow{\;\mathsf{d}\otimes 1\;} & SA \otimes A \otimes A \\
\Big\downarrow{\scriptstyle \mathsf{d}} & & & & \Big\downarrow{\scriptstyle 1\otimes\sigma} \\
SA \otimes A & & \xrightarrow[\;\mathsf{d}\otimes 1\;]{} & & SA \otimes A \otimes A
\end{array}
$$

## Example

For a polynomial $p(x_1, \ldots, x_n)$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\frac{\partial p}{\partial x_i}}{\partial x_j}(x_1, \ldots, x_n) \otimes x_j \otimes x_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\frac{\partial p}{\partial x_i}}{\partial x_j}(x_1, \ldots, x_n) \otimes x_i \otimes x_j$$

# Deriving Transformation - Definiton



# Example: Polynomials

## Example

$\mathsf{VEC}_{\mathbb{K}}$ is a differential category, with differential modality Sym and deriving transformation given by polynomial differentiation:

$$\mathsf{d} : \mathbb{K}[X] \to \mathbb{K}[X] \otimes V$$
$$\mathsf{p}(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i$$

- This example generalizes to modules over any commutative semiring.

# Example: Polynomials

## Example

$\mathsf{VEC}_{\mathbb{K}}$ is a differential category, with differential modality Sym and deriving transformation given by polynomial differentiation:

$$\mathsf{d} : \mathbb{K}[X] \to \mathbb{K}[X] \otimes V$$

$$\mathsf{p}(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} \frac{\partial \mathsf{p}}{\partial x_i}(x_1, \ldots, x_n) \otimes x_i$$

- This example generalizes to modules over any commutative semiring.

- In fact, the free commutative monoid monad (if it exists) on an additive symmetric monoidal category is always a differential modality:


Lemay, J.-S. P. **Coderelictions for Free Exponential Modalities**. (2021)

Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)


- This gives us lots of examples, such as the category of sets and relations, where the differential modality is given by finite bags, or the opposite category of modules, where the differential modality is given by the cofree cocommutative coalgebra.

# Example: Smooth Functions

## Example

Recall that a $\mathcal{C}^\infty$-**ring** is commutative $\mathbb{R}$-algebra $A$ such that for each for smooth map $f : \mathbb{R}^n \to \mathbb{R}$ there is a function $\Phi_f : A^n \to A$ and such that the $\Phi_f$ satisfy certain coherences between them.

**Ex.** For a smooth manifold $M$, $\mathcal{C}^\infty(M) = \{f : M \to \mathbb{R} | \ f \text{ smooth}\}$ is a $\mathcal{C}^\infty$-ring.

For every $\mathbb{R}$-vector space $V$, there is a free $\mathcal{C}^\infty$-ring over $V$, $\mathsf{S}^\infty(V)$. This induces a differential modality on $\mathsf{VEC}_\mathbb{R}$. In particular, $\mathsf{S}^\infty(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)$, and the deriving transformation is given by the usual differentiating of smooth functions:

$$\mathsf{d} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$$

$$f \longmapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes x_i$$

Cruttwell, G.S.H., Lemay, J.-S. P. and Lucyshyn-Wright, R.B.B. **Integral and differential structure on the free $C^\infty$-ring modality**. (2019)

# Other Examples

## Example

Other examples can be found in:

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

which in particular includes:

# Other Examples

## Example

Other examples can be found in:

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

which in particular includes:

- The free Rota-Baxter monad is a differential modality

# Other Examples

## Example

Other examples can be found in:

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

which in particular includes:

- The free Rota-Baxter monad is a differential modality
- The exterior algebra monad on finite dimensional $\mathbb{Z}_2$-vector spaces

# Other Examples

## Example

Other examples can be found in:

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

which in particular includes:

- The free Rota-Baxter monad is a differential modality
- The exterior algebra monad on finite dimensional $\mathbb{Z}_2$-vector spaces
- Fun fact: the free differential algebra monad is **NOT** a differential modality!

# Other Examples

## Example

Other examples can be found in:

Blute, R., Cockett, R., Seely, R.A.G., Lemay, J.-S. P. **Differential categories revisited**. (2019)

which in particular includes:

- The free Rota-Baxter monad is a differential modality
- The exterior algebra monad on finite dimensional $\mathbb{Z}_2$-vector spaces
- Fun fact: the free differential algebra monad is **NOT** a differential modality!

## Example

Every categorical model of Differential Linear Logic gives a (coalgebraic) differential category.

Fiore, M. **Differential structure in models of multiplicative biadditive intuitionistic linear logic** (2007)

Important examples include:

- Fininiteness Spaces, Köthe spaces, etc.

  Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)
- Convenient vector spaces

  Blute, R., Ehrhard, T. and Tasson, C. **A convenient differential category** (2012)

# A Quick Word about Tangent Categories and CDC

Since a differential modality is a monad, we can ask what can we say about its Kleisli category and its Eilenberg-Moore category?

# A Quick Word about Tangent Categories and CDC

Since a differential modality is a monad, we can ask what can we say about its Kleisli category and its Eilenberg-Moore category?

Well if you recall my map of differential categories I said that:

- The (opposite) Eilenberg-Moore category of a differential modality is a **tangent category**:

  R. Cockett, R., Lemay, J-S. P., Lucyshyn-Wright, R. **Tangent Categories from the Coalgebras of Differential Categories.** (2020)

  which means that in a way, we can think of algebras of a differential modality as "abstract smooth manifolds" or "abstract affine schemes".

  We'll actually talk about more algebras in a few slides...

- The opposite category of the Kleisli category of a differential modality is a **Cartesian differential category**:

  Blute, R., Cockett, R., Seely, R.A.G. **Cartesian Differential Categories** (2009)

  which means that in a way, we can think of the Kleisli category as a model of differential calculus over "abstract Euclidean spaces", and so we think of Kleisli maps as smooth maps (this is the fundamental idea in Differential Linear Logic).

# Things we can do in differential categories

# Things we can do in differential categories

- Derivations and Kähler differentials

Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)

# Things we can do in differential categories

- Derivations and Kähler differentials

Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)

- Hochschild complex, de Rham complex, and (co)homology

O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)

# Things we can do in differential categories

- Derivations and Kähler differentials

  Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)

- Hochschild complex, de Rham complex, and (co)homology

  O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)

- Differential algebras

  Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)

# Things we can do in differential categories

- Derivations and Kähler differentials
  - Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)
- Hochschild complex, de Rham complex, and (co)homology
  - O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)
- Differential algebras
  - Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)
- Antiderivatives, integration, and Taylor Series
  - Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)
  - Cockett, R., Lemay, J.-S.P. **Integral Categories and Calculus Categories**. (2018)
  - Lemay, J.-S.P. **Convenient Antiderivates for Differential Linear Categories**. (2020)
  - Lemay, J.-S.P. **An Ultrametric for Cartesian Differential Categories for Taylor Series Convergence**. (2024)

# Things we can do in differential categories

- Derivations and Kähler differentials
  - Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)
- Hochschild complex, de Rham complex, and (co)homology
  - O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)
- Differential algebras
  - Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)
- Antiderivatives, integration, and Taylor Series
  - Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)
  - Cockett, R., Lemay, J.-S.P. **Integral Categories and Calculus Categories**. (2018)
  - Lemay, J.-S.P. **Convenient Antiderivates for Differential Linear Categories**. (2020)
  - Lemay, J.-S.P. **An Ultrametric for Cartesian Differential Categories for Taylor Series Convergence**. (2024)
- Exponential Functions and Laplace Transforms:
  - Lemay, J.-S.P. **Exponential Functions for Cartesian Differential Categories**. (2018)
  - Kerjean, M., Lemay, J.-S.P. **Laplace Distributors and Laplace Transformations for Differential Categories**. (2024)

# Things we can do in differential categories

- Derivations and Kähler differentials

Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)

- Hochschild complex, de Rham complex, and (co)homology

O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)

- Differential algebras

Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)

- Antiderivatives, integration, and Taylor Series

Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)

Cockett, R., Lemay, J.-S.P. **Integral Categories and Calculus Categories**. (2018)

Lemay, J.-S.P. **Convenient Antiderivates for Differential Linear Categories**. (2020)

Lemay, J.-S.P. **An Ultrametric for Cartesian Differential Categories for Taylor Series Convergence**. (2024)

- Exponential Functions and Laplace Transforms:

Lemay, J.-S.P. **Exponential Functions for Cartesian Differential Categories**. (2018)

Kerjean, M., Lemay, J.-S.P. **Laplace Distributors and Laplace Transformations for Differential Categories**. (2024)

- Reverse differentiation:

Cruttwell, G., Gallagher, P., Lemay, J-S. P., Pronk, D. **Monoidal Reverse Differential Categories**. (2023)

# Things we can do in differential categories

- **Derivations** and Kähler differentials
  - Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)
- Hochschild complex, de Rham complex, and (co)homology
  - O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)
- Differential algebras
  - Lemay, J.-S.P. **Differential algebras in codifferential categories**. (2019)
- Antiderivatives, integration, and Taylor Series
  - Ehrhard, T. **An introduction to differential linear logic: proof-nets, models and antiderivatives**. (2018)
  - Cockett, R., Lemay, J.-S.P. **Integral Categories and Calculus Categories**. (2018)
  - Lemay, J.-S.P. **Convenient Antiderivates for Differential Linear Categories**. (2020)
  - Lemay, J.-S.P. **An Ultrametric for Cartesian Differential Categories for Taylor Series Convergence**. (2024)
- Exponential Functions and Laplace Transforms:
  - Lemay, J.-S.P. **Exponential Functions for Cartesian Differential Categories**. (2018)
  - Kerjean, M., Lemay, J.-S.P. **Laplace Distributors and Laplace Transformations for Differential Categories**. (2024)
- Reverse differentiation:
  - Cruttwell, G., Gallagher, P., Lemay, J-S. P., Pronk, D. **Monoidal Reverse Differential Categories**. (2023)

# Derivations

Before we can talk about differential graded algebras in a differential category, it will be helpful to first understand derivations in a differential category [5]

[5] Indeed recall that in a DGA, the differential $A_0 \to A_1$ is a derivation – we'll revisit this later.

# Derivations

Before we can talk about differential graded algebras in a differential category, it will be helpful to first understand derivations in a differential category [5]

Let $A$ be a commutative $\mathbb{K}$-algebra and $M$ be an $A$-module.

Then a **derivation** is a $\mathbb{K}$-linear map $\mathsf{D} : A \to M$ which satisfies the Leibniz rule:

$$\mathsf{D}(ab) = a\mathsf{D}(b) + b\mathsf{D}(a)$$

and from this it follows that $\mathsf{D}(1) = 0$.

[5] Indeed recall that in a DGA, the differential $A_0 \to A_1$ is a derivation – we'll revisit this later.

Before we can talk about differential graded algebras in a differential category, it will be helpful to first understand derivations in a differential category [5]

Let $A$ be a commutative $\mathbb{K}$-algebra and $M$ be an $A$-module.

Then a **derivation** is a $\mathbb{K}$-linear map $\mathsf{D} : A \to M$ which satisfies the Leibniz rule:

$$\mathsf{D}(ab) = a\mathsf{D}(b) + b\mathsf{D}(a)$$

and from this it follows that $\mathsf{D}(1) = 0$.

So if we wish to generalize derivations in a differential category we need to address three things:

- Algebra
- Module
- Derivation

[5] Indeed recall that in a DGA, the differential $A_0 \to A_1$ is a derivation – we'll revisit this later.

# Algebras of an Algebra Modality

Of course in a symmetric monoidal category, we can just take commutative monoids to play the role of our algebra...

# Algebras of an Algebra Modality

Of course in a symmetric monoidal category, we can just take commutative monoids to play the role of our algebra... but with an algebra modality we can do better!

# Algebras of an Algebra Modality

Of course in a symmetric monoidal category, we can just take commutative monoids to play the role of our algebra... but with an algebra modality we can do better!

For an algebra modality S, every S-algebra $(A, \nu : \mathsf{S}A \to A)$ admits a canonical commutative monoid structure where $\mathsf{m}^\nu : A \otimes A \to A$ and $\mathsf{u}^\nu : I \to A$ are defined as follows:

$$\mathsf{m}^\nu : A \otimes A \xrightarrow{\eta \otimes \eta} \mathsf{S}A \otimes \mathsf{S}A \xrightarrow{\mathsf{m}} \mathsf{S}A \xrightarrow{\nu} A$$

$$\mathsf{u}^\nu : I \xrightarrow{\mathsf{u}} \mathsf{S}A \xrightarrow{\nu} A$$

When applying this construction to a free S-algebra $(\mathsf{S}A, \mu)$ we get back $\mathsf{m}^\mu = \mathsf{m}$ and $\mathsf{u}^\mu = \mathsf{u}$.

# Algebras of an Algebra Modality

Of course in a symmetric monoidal category, we can just take commutative monoids to play the role of our algebra... but with an algebra modality we can do better!

For an algebra modality $\mathsf{S}$, every $\mathsf{S}$-algebra $(A, \nu : \mathsf{S}A \to A)$ admits a canonical commutative monoid structure where $\mathsf{m}^\nu : A \otimes A \to A$ and $\mathsf{u}^\nu : I \to A$ are defined as follows:

$$\mathsf{m}^\nu : A \otimes A \xrightarrow{\eta \otimes \eta} \mathsf{S}A \otimes \mathsf{S}A \xrightarrow{\mathsf{m}} \mathsf{S}A \xrightarrow{\nu} A$$

$$\mathsf{u}^\nu : I \xrightarrow{\mathsf{u}} \mathsf{S}A \xrightarrow{\nu} A$$

When applying this construction to a free $\mathsf{S}$-algebra $(\mathsf{S}A, \mu)$ we get back $\mathsf{m}^\mu = \mathsf{m}$ and $\mathsf{u}^\mu = \mathsf{u}$.

## Example

The Sym-algebras are precisely commutative $\mathbb{K}$-algebras, and the above monoid structure captures precisely the $\mathbb{K}$-algebra structure.

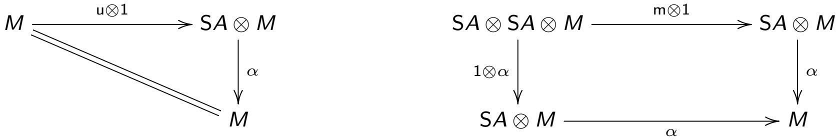
## Example

Recall that every $\mathcal{C}^\infty$-ring is an $\mathbb{R}$-algebra where the multiplication is given by $\Phi_m : A \times A \to A$, where $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the usual multiplication, which is a smooth function.

Then $\mathsf{S}^\infty$-algebras are precisely $\mathcal{C}^\infty$-rings, and the above monoid structure captures precisely the $\mathbb{R}$-algebra structure of a $\mathcal{C}^\infty$-ring.

# Modules?

Recall that for a commutative monoid $(A, \mathrm{m}, \mathrm{u})$, a $(A, \mathrm{m}, \mathrm{u})$-module is an object $M$ with an action $\alpha : A \otimes M \to A$:




Then for an S-algebra $(A, \nu)$, by an $(A, \nu)$-module we mean a $(A, \mathrm{m}^{\nu}, \mathrm{u}^{\nu})$-module.

## Example

This recaptures modules of commutative $\mathbb{K}$-algebras.

## Example

By modules for a $\mathcal{C}^{\infty}$-ring, they just mean a module over its underlying $\mathbb{R}$-algebra.

# Derivation for Differential Modalities

For a differential modality S, derivations are not just axiomatized by the Leibniz, but in fact by the chain rule which makes use of the deriving transformation!

# Derivation for Differential Modalities

For a differential modality S, derivations are not just axiomatized by the Leibniz, but in fact by the chain rule which makes use of the deriving transformation!

## Definition

Let $(A, \nu)$ be an S-algebra and $(M, \alpha)$ be a $(A, \nu)$-module. Then an S-**derivation** is a map $\mathsf{D} : A \to M$ such that the following diagram commutes:

$$\begin{array}{ccccc} SA & & \xrightarrow{\quad \nu \quad} & & A \\ \downarrow{\scriptstyle \mathsf{d}} & & & & \downarrow{\scriptstyle \mathsf{D}} \\ SA \otimes A & \xrightarrow[\nu \otimes \mathsf{D}]{} & A \otimes M & \xrightarrow[\alpha]{} & M \end{array}$$

$R.Blute$

$R.Lucyshyn - Wright$

$K.O'Neill$

Blute, R., Lucyshyn-Wright, R.B.B. and O'Neill, K. **Derivations in codifferential categories**. (2016)

The rough idea is that if $f \in SA$ is a smooth function $A \to I$ at which we can evaluate at $a \in A$:

$$\mathsf{D}(f(a)) = f'(a)\mathsf{D}(a)$$

# S-Derivation are Derivations

Every S-derivation $\mathrm{D} : A \to M$ satisfies the Leibniz Rule and the constant rule as well:

$$\begin{array}{ccc} I & \xrightarrow{\mathrm{u}^{\nu}} & A \\ & \searrow^{0} & \downarrow{\scriptstyle \mathrm{D}} \\ & & M \end{array}$$

$$\begin{array}{ccc} A \otimes A & \xrightarrow{(1\otimes \mathrm{D})+\sigma\circ(\mathrm{D}\otimes 1)} & A \otimes M \\ {\scriptstyle \mathrm{m}^{\nu}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ A & \xrightarrow[\mathrm{D}]{} & M \end{array}$$

# Examples

## Example

Sym-derivations correspond to usual derivations.

# Examples

## Example

Sym-derivations correspond to usual derivations. But wait I said Sym-derivations were axiomatized by the chain rule while a derivation is just axiomatized by the Leibniz rule...

# Examples

## Example

Sym-derivations correspond to usual derivations. But wait I said Sym-derivations were axiomatized by the chain rule while a derivation is just axiomatized by the Leibniz rule... well a derivation $\mathrm{D} : A \to M$ satisfies a chain rule for polynomials! So given a polynomial $p(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$, we have that:

$$\mathrm{D}\left(p(a_1, \ldots, a_n)\right) = \sum_{i=0}^{n} \frac{\partial p}{\mathrm{d}x_i}(a_1, \ldots, a_n)\mathrm{D}(a_i)$$

# Examples

## Example

Sym-derivations correspond to usual derivations. But wait I said Sym-derivations were axiomatized by the chain rule while a derivation is just axiomatized by the Leibniz rule... well a derivation $\mathrm{D} : A \to M$ satisfies a chain rule for polynomials! So given a polynomial $p(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$, we have that:

$$\mathrm{D}\left(p(a_1, \ldots, a_n)\right) = \sum_{i=0}^{n} \frac{\partial p}{\mathrm{d}x_i}(a_1, \ldots, a_n)\mathrm{D}(a_i)$$

## Example

$\mathrm{S}^\infty$-derivations correspond to $\mathcal{C}^\infty$-derivations:

E. Dubuc, A. Kock, **On 1-form classifiers** (1984)

which is an $\mathbb{R}$-linear map $\mathrm{D} : A \to M$ which satisfies the following chain rule for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$ (recall $\Phi_f : A^n \to A$):

$$\mathrm{D}\left(\Phi_f(a_1, \ldots, a_n)\right) = \sum_{i=0}^{n} \Phi_{\frac{\partial f}{\mathrm{d}x_i}}(a_1, \ldots, a_n)\mathrm{D}(a_i)$$

# Quick Word about Universal S-Derivations

The deriving transformation $\mathsf{d} : \mathsf{S}A \to \mathsf{S}A \otimes A$ is an S-derivation between the free S-algebra $(\mathsf{S}A, \mu)$ and its module $(\mathsf{S}A \otimes A, \mathsf{m} \otimes 1)$, where the S-Derivation axiom is precisely the deriving transformation chain rule axiom.

# Quick Word about Universal S-Derivations

The deriving transformation $\mathsf{d} : \mathsf{S}A \to \mathsf{S}A \otimes A$ is an S-derivation between the free S-algebra $(\mathsf{S}A, \mu)$ and its module $(\mathsf{S}A \otimes A, \mathsf{m} \otimes 1)$, where the S-Derivation axiom is precisely the deriving transformation chain rule axiom.

In fact $\mathsf{d} : \mathsf{S}A \to \mathsf{S}A \otimes A$ is the universal S-derivation for $(\mathsf{S}A, \mu)$, in the sense that all S-derivations for $(\mathsf{S}A, \mu)$ factor through $\mathsf{d} : \mathsf{S}A \to \mathsf{S}A \otimes A$.

# Quick Word about Universal S-Derivations

The deriving transformation $\mathsf{d} : SA \to SA \otimes A$ is an S-derivation between the free S-algebra $(SA, \mu)$ and its module $(SA \otimes A, \mathsf{m} \otimes 1)$, where the S-Derivation axiom is precisely the deriving transformation chain rule axiom.

In fact $\mathsf{d} : SA \to SA \otimes A$ is the universal S-derivation for $(SA, \mu)$, in the sense that all S-derivations for $(SA, \mu)$ factor through $\mathsf{d} : SA \to SA \otimes A$.

In fact, with enough coequalizers, we can build the generalizations of Kähler differentials $\Omega^{\bullet}_{(A,nu)}$ for all S-algebras $(A, \nu)$, that is, we can build universal S-derivations for S-algebras.

Blute, R., Lucyshyn-Wright, R.B.B. and O'NeilL, K. **Derivations in codifferential categories**. (2016)

From here we can study the de Rham complex, Hochschild complex, and (co)homology:

O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)

# Differential Graded Algebras

For us, our differential graded algebras will be commutative and unital, and $\mathbb{N}$-graded.

# Differential Graded Algebras

For us, our differential graded algebras will be commutative and unital, and $\mathbb{N}$-graded.

Over a field $\mathbb{K}$, recall that a differential graded algebra is a:

# Differential Graded Algebras

For us, our differential graded algebras will be commutative and unital, and $\mathbb{N}$-graded.

Over a field $\mathbb{K}$, recall that a differential graded algebra is a:

- (commutative and unital) graded $\mathbb{K}$-algebra $A^\bullet = \bigoplus_{n=0}^{\infty} A^n$, so we have that:

$$A^m A^n \subseteq A^{m+n} \qquad \deg(ab) = \deg(a) + \deg(b) \qquad \deg(1) = 0 \qquad ab = (-1)^{\deg(a)\deg(b)} ba$$

# Differential Graded Algebras

For us, our differential graded algebras will be commutative and unital, and $\mathbb{N}$-graded.

Over a field $\mathbb{K}$, recall that a differential graded algebra is a:

- (commutative and unital) graded $\mathbb{K}$-algebra $A^\bullet = \bigoplus_{n=0}^{\infty} A^n$, so we have that:

$$A^m A^n \subseteq A^{m+n} \qquad \deg(ab) = \deg(a) + \deg(b) \qquad \deg(1) = 0 \qquad ab = (-1)^{\deg(a)\deg(b)} ba$$

- With differentials $\partial : A^n \to A^{n+1}$ such that:

$$\partial \circ \partial = 0 \qquad\qquad \partial(ab) = \partial(a)b + (-1)^{\deg(a)} a\partial(b)$$

## Example

For any $\mathbb{K}$-algebra $A$, its de Rham complex $\Omega(A)^\bullet = \mathrm{Ext}_A(\Omega(A))$ is a differential graded algebra where:

$$\Omega^n(A) = \Omega(A) \wedge \ldots \wedge \Omega(A) \qquad \partial\left(a_0 \mathrm{d}(a_1) \wedge \ldots \wedge \mathrm{d}(a_n)\right) = \mathrm{d}(a_0) \wedge \mathrm{d}(a_1) \wedge \ldots \wedge \mathrm{d}(a_n)$$

Similarly, for any smooth manifold $M$, its de Rham complex $\Omega(M)^\bullet$ of differential forms is a differential graded algebra.

# Differential Graded Algebras

We can generalize differential graded algebras in an additive symmetric monoidal category **with negatives**. So differential graded algebra $A^\bullet$ consists of:

# Differential Graded Algebras

We can generalize differential graded algebras in an additive symmetric monoidal category **with negatives**. So differential graded algebra $A^\bullet$ consists of:

- A family of objects $A^n$ for every $n \in \mathbb{N}$;

# Differential Graded Algebras

We can generalize differential graded algebras in an additive symmetric monoidal category **with negatives**. So differential graded algebra $A^\bullet$ consists of:

- A family of objects $A^n$ for every $n \in \mathbb{N}$;
- With maps $u_0 : I \to A^0$ and $m_{m,n} : A^m \otimes A^n \to A^{m+n}$ such that:

$$
\begin{array}{ccc}
A^n & \xrightarrow{u_0 \otimes 1} & A^0 \otimes A^n \\
{\scriptstyle 1\otimes u_0}\downarrow & \searrow^{=} & \downarrow{\scriptstyle m_{0,n}} \\
A^n \otimes A^0 & \xrightarrow[m_{n,0}]{} & A^n
\end{array}
\qquad
\begin{array}{ccc}
A^m \otimes A^n \otimes A^p & \xrightarrow{m_{m,n}\otimes 1} & A^m \otimes A^{n+p} \\
{\scriptstyle 1\otimes m_{n,p}}\downarrow & & \downarrow{\scriptstyle m_{m,n+p}} \\
A^{m+n} \otimes A^p & \xrightarrow[m_{m+n,p}]{} & A^{m+n+p}
\end{array}
\qquad
\begin{array}{ccc}
A^m \otimes A^n & \xrightarrow{\sigma} & A^n \otimes A^m \\
 & {\scriptstyle m_{m,n}}\searrow & \downarrow{\scriptstyle (-1)^{mn} m_{n,m}} \\
 & & A^{m+n}
\end{array}
$$

# Differential Graded Algebras

We can generalize differential graded algebras in an additive symmetric monoidal category **with negatives**. So differential graded algebra $A^\bullet$ consists of:

- A family of objects $A^n$ for every $n \in \mathbb{N}$;
- With maps $\mathrm{u}_0 : I \to A^0$ and $\mathrm{m}_{m,n} : A^m \otimes A^n \to A^{m+n}$ such that:

$$
\begin{array}{ccc}
A^n & \xrightarrow{\mathrm{u}_0 \otimes 1} & A^0 \otimes A^n \\
{\scriptstyle 1 \otimes \mathrm{u}_0}\downarrow & \searrow\!\!\!= & \downarrow{\scriptstyle \mathrm{m}_{0,n}} \\
A^n \otimes A^0 & \xrightarrow[\mathrm{m}_{n,0}]{} & A^n
\end{array}
\qquad
\begin{array}{ccc}
A^m \otimes A^n \otimes A^p & \xrightarrow{\mathrm{m}_{m,n} \otimes 1} & A^m \otimes A^{n+p} \\
{\scriptstyle 1 \otimes \mathrm{m}_{n,p}}\downarrow & & \downarrow{\scriptstyle \mathrm{m}_{m,n+p}} \\
A^{m+n} \otimes A^p & \xrightarrow[\mathrm{m}_{m+n,p}]{} & A^{m+n+p}
\end{array}
\qquad
\begin{array}{ccc}
A^m \otimes A^n & \xrightarrow{\sigma} & A^n \otimes A^m \\
& {\scriptstyle \mathrm{m}_{m,n}}\searrow & \downarrow{\scriptstyle (-1)^{mn}\mathrm{m}_{n,m}} \\
& & A^{m+n}
\end{array}
$$

- With maps $\partial : A^n \to A^{n+1}$ such that:

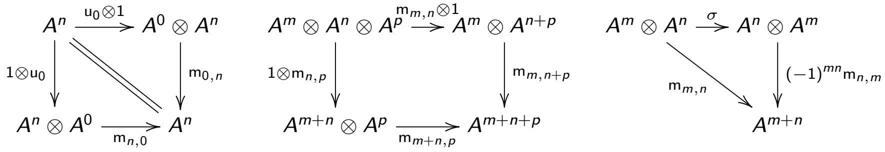
$$\partial \circ \partial = 0$$

$$\partial \circ \mathrm{m}_{m,n} = \mathrm{m}_{m+1,n} \circ (\partial \otimes 1) + ((-1)^m \mathrm{m}_{m,n+1}) \circ (1 \otimes \partial)$$

# Differential Graded Algebras

We can generalize differential graded algebras in an additive symmetric monoidal category **with negatives**. So differential graded algebra $A^\bullet$ consists of:

- A family of objects $A^n$ for every $n \in \mathbb{N}$;
- With maps $u_0 : I \to A^0$ and $m_{m,n} : A^m \otimes A^n \to A^{m+n}$ such that:




- With maps $\partial : A^n \to A^{n+1}$ such that:

$$\partial \circ \partial = 0$$

$$\partial \circ m_{m,n} = m_{m+1,n} \circ (\partial \otimes 1) + ((-1)^m m_{m,n+1}) \circ (1 \otimes \partial)$$

So how can we improve upon this using our differential modality S?

Notice that in our differential graded algebra:

- $A^0$ is a commutative monoid;
- $A^1$ is a $A^0$-module
- $\partial : A^0 \to A^1$ is a derivation

# Differential Graded S-Algebras

Notice that in our differential graded algebra:

- $A^0$ is a commutative monoid;
- $A^1$ is a $A^0$-module
- $\partial : A^0 \to A^1$ is a derivation

So for our differential graded S-algebras, we essentially simply ask that:

- $A^0$ is an **S-algebra**;
- $A^1$ is a $A^0$-module
- $\partial : A^0 \to A^1$ is an **S-derivation**

So you can just ask for this extra structure, and we can also be slightly more compact...

# Differential Graded S-Algebras

In a differential category **with negatives**, a differential graded S-algebra $A^{\bullet}$ consists of:

# Differential Graded S-Algebras

In a differential category **with negatives**, a differential graded S-algebra $A^\bullet$ consists of:

- A family of objects $A^n$ for every $n \in \mathbb{N}$;
- With a map $\nu : \mathsf{S}A^0 \to A^0$ such that $(A^0, \nu)$ is an S-algebra
- With maps $\mathsf{m}_{m,n} : A^m \otimes A^n \to A^{m+n}$, for $m \geq 1$ or $n \geq 1$, such that with $\mathsf{u}^\nu$ and $\mathsf{m}_{0,0} = \mathsf{m}^\nu$ the diagrams from the previous slide commutes.
- With maps $\partial : A^n \to A^{n+1}$ such that:

$$\partial \circ \partial = 0$$

$$\partial \circ \mathsf{m}_{0,n} \circ (\nu \otimes 1) = \mathsf{m}_{0,n+1} \circ (\nu \otimes \partial) + \mathsf{m}_{0,n+1} \circ (\nu \otimes \mathsf{m}_{1,n}) \circ (1 \otimes \partial \otimes 1) \circ (\mathsf{d} \otimes 1)$$

This last equation is a higher-level chain rule, notice that the deriving transformation d appears, and when taking $n = 0$ and inserting a unit, we get precisely that $\partial : A^0 \to A^1$ is an S-derivation.

# Differential Graded S-Algebras

In a differential category **with negatives**, a differential graded S-algebra $A^\bullet$ consists of:

- A family of objects $A^n$ for every $n \in \mathbb{N}$;
- With a map $\nu : \mathsf{S}A^0 \to A^0$ such that $(A^0, \nu)$ is an S-algebra
- With maps $\mathsf{m}_{m,n} : A^m \otimes A^n \to A^{m+n}$, for $m \geq 1$ or $n \geq 1$, such that with $\mathsf{u}^\nu$ and $\mathsf{m}_{0,0} = \mathsf{m}^\nu$ the diagrams from the previous slide commutes.
- With maps $\partial : A^n \to A^{n+1}$ such that:

$$\partial \circ \partial = 0$$

$$\partial \circ \mathsf{m}_{0,n} \circ (\nu \otimes 1) = \mathsf{m}_{0,n+1} \circ (\nu \otimes \partial) + \mathsf{m}_{0,n+1} \circ (\nu \otimes \mathsf{m}_{1,n}) \circ (1 \otimes \partial \otimes 1) \circ (\mathsf{d} \otimes 1)$$

This last equation is a higher-level chain rule, notice that the deriving transformation d appears, and when taking $n = 0$ and inserting a unit, we get precisely that $\partial : A^0 \to A^1$ is an S-derivation.

However it turns out these higher-level chain rules are not needed!

# Differential Graded S-Algebras – Another Definition

## Lemma

*To give a differential graded S-algebra is equivalent to giving a differential graded algebra* $A^\bullet$ *such that:*

- *There is a map* $\nu : \mathsf{S}A^0 \to A^0$ *such that* $(A^0, \nu)$ *is an S-algebra;*
- $\nu : (\mathsf{S}A^0, \mathrm{m}, \mathrm{u}) \to (A^0, \mathrm{m}_{0,0}, \mathrm{u}_0)$ *is a monoid morphism;*
- $\partial : A^0 \to A^1$ *is an S-derivation*

## Proof.

Asking that $\nu$ is a monoid morphism forces that $\mathrm{u}_0 = \mathrm{u}^\nu$ and $\mathrm{m}_{0,0} = \mathrm{m}^\nu$. While the higher-level chain rule = chain rule at level 1 + higher-level Leibniz rules. □

**Example**

Differential graded Sym-algebras correspond precisely to (commutative and unital) differential graded $\mathbb{K}$-algebras.

# Examples

## Example

Differential graded Sym-algebras correspond precisely to (commutative and unital) differential graded $\mathbb{K}$-algebras.

## Example

Differential graded $\mathsf{S}^{\infty}$-algebras correspond precisely to differential graded $\mathcal{C}^{\infty}$-rings in the sense of the nlab article: "smooth differential forms form the free $\mathcal{C}^{\infty}$-DGA on smooth functions" (LINK) written by Dmitri Pavlov:

A differential graded $\mathcal{C}^{\infty}$-ring is precisely a differential graded $\mathbb{R}$-algebra such that $A^0$ is a $\mathcal{C}^{\infty}$-ring and $\partial : A^0 \to A^1$ is a $\mathcal{C}^{\infty}$-derivation. The earliest mention of differential graded $\mathcal{C}^{\infty}$-rings in this sense appears briefly in a remark in:


Domenico Fiorenza, Urs Schreiber, Jim Stasheff, **Čech cocycles for differential characteristic classes: an $\infty$-Lie theoretic construction** (2012)

# Justifying our definition...

So when we first came up with the definition of an differential graded S-algebra, we were a bit worried it was a bit ad hoc...

# Justifying our definition...

So when we first came up with the definition of an differential graded S-algebra, we were a bit worried it was a bit ad hoc...

However when we found this $\mathcal{C}^\infty$-ring example, this seemed to justify our definition somewhat.

# Justifying our definition...

So when we first came up with the definition of an differential graded S-algebra, we were a bit worried it was a bit ad hoc...

However when we found this $\mathcal{C}^\infty$-ring example, this seemed to justify our definition somewhat.

But I wasn't satisfied: how could we really justify that we had the right definition?

# Justifying our definition...

So when we first came up with the definition of an differential graded S-algebra, we were a bit worried it was a bit ad hoc...

However when we found this $\mathcal{C}^{\infty}$-ring example, this seemed to justify our definition somewhat.

But I wasn't satisfied: how could we really justify that we had the right definition?

For this we turn to the well-known fact that:

$$\text{Differential Graded Algebras} \quad \equiv \quad \text{Commutative Monoids in Chain Complexes}$$

# Justifying our definition...

So when we first came up with the definition of an differential graded S-algebra, we were a bit worried it was a bit ad hoc...

However when we found this $\mathcal{C}^{\infty}$-ring example, this seemed to justify our definition somewhat.

But I wasn't satisfied: how could we really justify that we had the right definition?

For this we turn to the well-known fact that:

$$\text{Differential Graded Algebras} \qquad \equiv \qquad \text{Commutative Monoids in Chain Complexes}$$

Of course we have our differential modality S, to justify our definition we want:

$$\text{Differential Graded S-Algebras} \qquad \equiv \qquad \text{S-Algebras in Chain Complexes}$$

# Main Theorem

**Theorem**

*In a (algebraic) differential category with negatives and enough colimits:*

- *There is a differential modality* $\overline{\mathsf{S}}$ *on the category of chain complexes (which is in fact a lifting of the differential modality* $\mathsf{S}$ *of the base category).*
- *Differential graded* $\mathsf{S}$*-algebras correspond precisely to* $\overline{\mathsf{S}}$*-algebras (which in particular are also commutative comonoids in chain complexes)*

The rest of the talk becomes more technical... but this is the slide to take away from today!

# Chain Complexes

Recall that for a category $\mathbb{X}$ with zero maps, we can build its category of ($\mathbb{N}$-indexed) chain complexes, $\mathsf{Ch}(\mathbb{X})$ where:

- Objects $A^\bullet$ are chain complexes, that is, a family of object $A^n$ with maps $\partial : A^n \to A^{n+1}$ such that $\partial \circ \partial = 0$.

$$\dots \xrightarrow{\quad\partial\quad} A^n \xrightarrow{\quad\partial\quad} A^{n+1} \xrightarrow{\quad\partial\quad} A^{n+2} \xrightarrow{\quad\partial\quad} \dots$$

- Maps $f^\bullet : A^\bullet \to B^\bullet$ is a family of maps $f^n : A^n \to B^n$ which commutes with the differentials, $f^n \circ \partial = \delta \circ f^n$.

# Chain Complexes

Recall that for a category $\mathbb{X}$ with zero maps, we can build its category of ($\mathbb{N}$-indexed) chain complexes, $\mathsf{Ch}(\mathbb{X})$ where:

- Objects $A^\bullet$ are chain complexes, that is, a family of object $A^n$ with maps $\partial : A^n \to A^{n+1}$ such that $\partial \circ \partial = 0$.

$$\ldots \xrightarrow{\quad\partial\quad} A^n \xrightarrow{\quad\partial\quad} A^{n+1} \xrightarrow{\quad\partial\quad} A^{n+2} \xrightarrow{\quad\partial\quad} \ldots$$

- Maps $f^\bullet : A^\bullet \to B^\bullet$ is a family of maps $f^n : A^n \to B^n$ which commutes with the differentials, $f^n \circ \partial = \delta \circ f^n$.

Now suppose that $\mathbb{X}$ is an additive symmetric monoidal category with negatives and **finite biproducts**. Then $\mathsf{Ch}(\mathbb{X})$ is a symmetric monoidal category where:

$$(A^\bullet \otimes B^\bullet)^n :\, = \bigoplus_{p+q=n} A^p \otimes B^q$$

and the differential is induced by this formula (abusing notation):

$$\partial \otimes 1 + (-1)^{\deg(A^\bullet)}(1 \otimes \delta)$$

This makes $\mathsf{Ch}(\mathbb{X})$ also n additive symmetric monoidal category with negatives (and finite biproducts).

# Differential Graded Algebras are Commutative Monoids

**Proposition**

*For an additive symmetric monoidal category $\mathbb{X}$ with negatives and finite biproducts, a commutative differential graded algebra in $\mathbb{X}$ (as given in previous slides) is precisely a commutative monoid in* $\mathsf{Ch}(\mathbb{X})$.

I gave the unpacked definition first since I didn't need biproducts to give the definition of a differential graded algebra, and since differential categories do not necessarily need biproducts (though having biproducts is desirable).

# Objective: Lifting

There is a "forgetful functor" $\mathsf{U}_0 : \mathsf{Ch}(\mathbb{X}) \to \mathbb{X}$ which forgets about your chain complex by picking out the zero component, $\mathsf{U}_0(A^\bullet)$.

# Objective: Lifting

There is a "forgetful functor" $\mathsf{U}_0 : \mathsf{Ch}(\mathbb{X}) \to \mathbb{X}$ which forgets about your chain complex by picking out the zero component, $\mathsf{U}_0(A^\bullet)$.

The goal now is if $\mathbb{X}$ is a differential category (with negatives and finite biproducts), we want to construct a **lifting** of its differential modality $\mathsf{S}$ to $\mathsf{Ch}(\mathbb{X})$, that is, we want to construct a differential modality $\overline{\mathsf{S}}$ on $\mathsf{Ch}(\mathbb{X})$ such that:

$$\begin{array}{ccc}
\mathsf{Ch}(\mathbb{X}) & \xrightarrow{\overline{\mathsf{S}}} & \mathsf{Ch}(\mathbb{X}) \\
{\scriptstyle \mathsf{U}_0}\downarrow & & \downarrow{\scriptstyle \mathsf{U}_0} \\
\mathbb{X} & \xrightarrow[\mathsf{S}]{} & \mathbb{X}
\end{array}$$

# Our First Attempt

To help give the construction of this lifting, let's look at our favourite example $\mathsf{S} = \mathsf{Sym}$.

# Our First Attempt

To help give the construction of this lifting, let's look at our favourite example $\mathsf{S} = \mathsf{Sym}$.

Now Sym is the free commutative monoid monad on $\mathsf{VEC}_{\mathbb{K}}$, so its lifting should be the free commutative monoid monad on $\mathsf{Ch}(\mathsf{VEC}_{\mathbb{K}})$. In other words, the lifting we are looking for is the free differential graded algebra construction on a chain complex.

# Our First Attempt

To help give the construction of this lifting, let's look at our favourite example $\mathsf{S} = \mathsf{Sym}$.

Now Sym is the free commutative monoid monad on $\mathsf{VEC}_{\mathbb{K}}$, so its lifting should be the free commutative monoid monad on $\mathsf{Ch}(\mathsf{VEC}_{\mathbb{K}})$. In other words, the lifting we are looking for is the free differential graded algebra construction on a chain complex.

So given a chain complex $V^{\bullet} = \bigoplus_{n=0}^{\infty} V^n$ (represented as an $\mathbb{N}$-graded $\mathbb{K}$-vector space), the free (commutative and unital) differential graded algebra over $V^{\bullet}$ is often presented as follows:

$$\overline{\mathsf{Sym}}(V^{\bullet}) = \mathsf{Sym}\left(\bigoplus_{\text{even}} V^n\right) \otimes \mathsf{Ext}\left(\bigoplus_{\text{odd}} V^n\right)$$

where Sym is the symmetric algebra and Ext is the exterior algebra.

# Our First Attempt

To help give the construction of this lifting, let's look at our favourite example $\mathsf{S} = \mathsf{Sym}$.

Now Sym is the free commutative monoid monad on $\mathsf{VEC}_{\mathbb{K}}$, so its lifting should be the free commutative monoid monad on $\mathsf{Ch}(\mathsf{VEC}_{\mathbb{K}})$. In other words, the lifting we are looking for is the free differential graded algebra construction on a chain complex.

So given a chain complex $V^{\bullet} = \bigoplus_{n=0}^{\infty} V^n$ (represented as an $\mathbb{N}$-graded $\mathbb{K}$-vector space), the free (commutative and unital) differential graded algebra over $V^{\bullet}$ is often presented as follows:

$$\overline{\mathsf{Sym}}(V^{\bullet}) = \mathsf{Sym}\left(\bigoplus_{\text{even}} V^n\right) \otimes \mathsf{Ext}\left(\bigoplus_{\text{odd}} V^n\right)$$

where Sym is the symmetric algebra and Ext is the exterior algebra.

So our first attempt was to describe our lifting as:

$$\overline{\mathsf{S}}(V^{\bullet}) = \mathsf{S}\left(\bigoplus_{\text{even}} V^n\right) \otimes \mathsf{Ext}\left(\bigoplus_{\text{odd}} V^n\right)$$

assuming we can build the exterior algebra somehow.

# Our First Attempt

To help give the construction of this lifting, let's look at our favourite example $\mathsf{S} = \mathsf{Sym}$.

Now Sym is the free commutative monoid monad on $\mathsf{VEC}_{\mathbb{K}}$, so its lifting should be the free commutative monoid monad on $\mathsf{Ch}(\mathsf{VEC}_{\mathbb{K}})$. In other words, the lifting we are looking for is the free differential graded algebra construction on a chain complex.

So given a chain complex $V^\bullet = \bigoplus_{n=0}^{\infty} V^n$ (represented as an $\mathbb{N}$-graded $\mathbb{K}$-vector space), the free (commutative and unital) differential graded algebra over $V^\bullet$ is often presented as follows:

$$\overline{\mathsf{Sym}}(V^\bullet) = \mathsf{Sym}\left(\bigoplus_{\text{even}} V^n\right) \otimes \mathsf{Ext}\left(\bigoplus_{\text{odd}} V^n\right)$$

where Sym is the symmetric algebra and Ext is the exterior algebra.

So our first attempt was to describe our lifting as:

$$\overline{\mathsf{S}}(V^\bullet) = \mathsf{S}\left(\bigoplus_{\text{even}} V^n\right) \otimes \mathsf{Ext}\left(\bigoplus_{\text{odd}} V^n\right)$$

assuming we can build the exterior algebra somehow. The problem is: (1) we might not have countable coproducts and (2) we couldn't figure out what the grading of this would be since S may not be nicely graded like Sym...

# Our First Attempt

To help give the construction of this lifting, let's look at our favourite example $\mathsf{S} = \mathsf{Sym}$.

Now Sym is the free commutative monoid monad on $\mathsf{VEC}_{\mathbb{K}}$, so its lifting should be the free commutative monoid monad on $\mathsf{Ch}(\mathsf{VEC}_{\mathbb{K}})$. In other words, the lifting we are looking for is the free differential graded algebra construction on a chain complex.

So given a chain complex $V^\bullet = \bigoplus_{n=0}^{\infty} V^n$ (represented as an $\mathbb{N}$-graded $\mathbb{K}$-vector space), the free (commutative and unital) differential graded algebra over $V^\bullet$ is often presented as follows:

$$\overline{\mathsf{Sym}}(V^\bullet) = \mathsf{Sym}\left(\bigoplus_{\text{even}} V^n\right) \otimes \mathsf{Ext}\left(\bigoplus_{\text{odd}} V^n\right)$$

where Sym is the symmetric algebra and Ext is the exterior algebra.

So our first attempt was to describe our lifting as:

$$\overline{\mathsf{S}}(V^\bullet) = \mathsf{S}\left(\bigoplus_{\text{even}} V^n\right) \otimes \mathsf{Ext}\left(\bigoplus_{\text{odd}} V^n\right)$$

assuming we can build the exterior algebra somehow. The problem is: (1) we might not have countable coproducts and (2) we couldn't figure out what the grading of this would be since S may not be nicely graded like Sym... But it turns out that the construction is very close to what we are searching for! We'll still need the following for our lifting construction:

- Symmetrized Powers
- Exterior Powers

# Symmetrized Powers and Exterior Powers

Recall that the $n$-th **symmetrized power** of $A$:

$$A^{\otimes_s^n}$$

is the coequalizer of all permutations $\omega : A^{\otimes^n} \to A^{\otimes^n}$.

While the $n$-th **exterior power** of $A$:

$$\bigwedge^n A$$

is the coequalizer of all permutations scalar multiplied by their sign $\mathrm{sgn}(\omega)\omega : A^{\otimes^n} \to A^{\otimes^n}$.

And we assume that $\otimes$ preserves these coequalizers.

# Symmetrized Powers and Exterior Powers

Recall that the $n$-th **symmetrized power** of $A$:

$$A^{\otimes_s^n}$$

is the coequalizer of all permutations $\omega : A^{\otimes^n} \to A^{\otimes^n}$.

While the $n$-th **exterior power** of $A$:

$$\bigwedge^n A$$

is the coequalizer of all permutations scalar multiplied by their sign $\mathrm{sgn}(\omega)\omega : A^{\otimes^n} \to A^{\otimes^n}$.

And we assume that $\otimes$ preserves these coequalizers.

However for our construction what we really want is to take the symmetrized/exterior powers of different objects:

$$X \otimes_s Y \qquad\qquad X \wedge Y$$

this is actually used in the grading of the free differential graded algebra.

# Symmetrized Powers and Exterior Powers[6]

So for a finite family of objects $A_1$, $A_2$, .., $A_n$:

- Their **symmetrized power** $A_1 \otimes_s \ldots \otimes_s A_n$ is defined as the coequalizer of all permutations on $(\bigoplus_{i=1}^{n} A_i)^{\otimes^n}$ pre-composed by the injection:

$$A_1 \otimes \ldots \otimes A_n \to (\bigoplus_{i=1}^{n} A_i) \otimes \ldots \otimes (\bigoplus_{i=1}^{n} A_i) \xrightarrow{\omega} (\bigoplus_{i=1}^{n} A_i) \otimes \ldots \otimes (\bigoplus_{i=1}^{n} A_i)$$

- Their **exterior power** $A_1 \wedge \ldots \wedge A_n$ is defined as the coequalizer of all permutations on $(\bigoplus_{i=1}^{n} A_i)^{\otimes^n}$ scalar multiplied by their sign pre-composed by the injection:

$$A_1 \otimes \ldots \otimes A_n \to (\bigoplus_{i=1}^{n} A_i) \otimes \ldots \otimes (\bigoplus_{i=1}^{n} A_i) \xrightarrow{\mathrm{sgn}(\omega)\omega} (\bigoplus_{i=1}^{n} A_i) \otimes \ldots \otimes (\bigoplus_{i=1}^{n} A_i)$$

We ask that $\otimes$ preserves these coequalizers and also notice that:

$$X \otimes_s Y = Y \otimes_s X \qquad X \wedge Y = Y \wedge X$$

[6] If anyone knows a reference for this, please let me know!

# Lifting Construction

## Theorem

*Let $\mathbb{X}$ be a differential category, with differential modality $\mathsf{S}$, and also with negatives, finite biproducts, symmetrized powers, and exterior powers. Then $\mathsf{Ch}(\mathbb{X})$ is a differential category with differential modality $\overline{\mathsf{S}}$ defined as follows:*

# Lifting Construction

## Theorem

*Let $\mathbb{X}$ be a differential category, with differential modality $\mathsf{S}$, and also with negatives, finite biproducts, symmetrized powers, and exterior powers. Then $\mathsf{Ch}(\mathbb{X})$ is a differential category with differential modality $\overline{\mathsf{S}}$ defined as follows:*

$$\overline{\mathsf{S}}\,(A^\bullet)^n = \bigoplus_{\substack{p_i \text{ even},\ q_j \text{ odd},\ p_i, q_j \geq 1 \\ \sum p_i + \sum q_j = n}} \mathsf{S}(A^0) \otimes (A^{p_1} \otimes_s \ldots \otimes_s A^{p_m}) \otimes (A^{q_1} \wedge \ldots \wedge A^{q_k})$$

*The differential involves both the differential of $A^\bullet$ and the deriving transformation $\mathsf{d}$ (we will see an example in a few slides).*

The idea is that homogenous elements is a monomial:

$$f(x)x_1x_2\ldots x_n$$

where $f(x) \in \mathsf{S}(A^0)$ is a smooth function and $x_i \in A^j$ for $j \geq 1$. And:

$$f(x)x_1x_2\ldots x_ix_{i+1}\ldots x_n = (-1)^{\deg(x_i)\deg(x_{i+1})} fx_1x_2\ldots x_{i+1}x_i\ldots x_n$$

And the differential is given by (where $\mathsf{d}(f) = f'\mathsf{d}x$):

$$\partial\,(f(x)x_1x_2\ldots x_n) = f'(x)\partial(\mathsf{d}x)x_1x_2\ldots x_n + \sum fx_1x_2\ldots\partial(x_i)\ldots x_n$$

# Lifting Construction

## Theorem

*Let $\mathbb{X}$ be a differential category, with differential modality $\mathsf{S}$, and also with negatives, finite biproducts, symmetrized powers, and exterior powers. Then $\mathsf{Ch}(\mathbb{X})$ is a differential category with differential modality $\overline{\mathsf{S}}$ defined as follows:*

$$\overline{\mathsf{S}}\,(A^\bullet)^n = \bigoplus_{\substack{p_i \text{ even, } q_j \text{ odd, } p_i, q_j \geq 1 \\ \sum p_i + \sum q_j = n}} \mathsf{S}(A^0) \otimes (A^{p_1} \otimes_s \ldots \otimes_s A^{p_m}) \otimes (A^{q_1} \wedge \ldots \wedge A^{q_k})$$

*The differential involves both the differential of $A^\bullet$ and the deriving transformation $\mathsf{d}$ (we will see an example in a few slides). Moreover the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{Ch}(\mathbb{X}) & \xrightarrow{\overline{\mathsf{S}}} & \mathsf{Ch}(\mathbb{X}) \\ {\scriptstyle \mathsf{U}_0}\downarrow & & \downarrow{\scriptstyle \mathsf{U}_0} \\ \mathbb{X} & \xrightarrow[\mathsf{S}]{} & \mathbb{X} \end{array}$$

# Example: This gives us back the Free DGA

## Example

For $\mathrm{VEC}_{\mathbb{K}}$, the lifting of Sym is precisely the free differential graded algebra as defined in the previous slide:

$$\overline{\mathrm{Sym}}(V^\bullet) = \mathrm{Sym}\left(\bigoplus_{\text{even}} V^n\right) \otimes \mathrm{Ext}\left(\bigoplus_{\text{odd}} V^n\right)$$

# Main Theorem

## Theorem

*$\overline{\mathsf{S}}$-algebras correspond precisely to differential graded* S*-algebras in* $\mathbb{X}$*.*

*So we have equivalence of categories between the Eilenberg-Moore category* $\mathsf{Alg}(\overline{\mathsf{S}})$ *and the category of differential graded* S*-algebras.*

## Corollary

*(The opposite category of) Differential graded* S*-algebras form a tangent category*[a]*.*

[a] In particular this means that commutative and unital differential graded algebras form a tangent category.

# Proof Sketch

To get an understanding of the proof we need to understand, we need to understand the degree 0 and 1 of $\overline{\mathsf{S}}(A^\bullet)^\bullet$ and the $\overline{\mathsf{S}}(A^\bullet)^0 \to \overline{\mathsf{S}}(A^\bullet)^1$.

# Proof Sketch

To get an understanding of the proof we need to understand, we need to understand the degree 0 and 1 of $\overline{\mathsf{S}}(A^\bullet)^\bullet$ and the $\overline{\mathsf{S}}(A^\bullet)^0 \to \overline{\mathsf{S}}(A^\bullet)^1$.

Degree 0 is simply $\overline{\mathsf{S}}(A^\bullet)^0 = \mathsf{S}(A^0)$.

Degree 1 is $\overline{\mathsf{S}}(A^\bullet)^1 = \mathsf{S}(A^0) \otimes A^1$

The differential is $\overline{\mathsf{S}}(A^\bullet)^0 \to \overline{\mathsf{S}}(A^\bullet)^1$:

$$\mathsf{S}(A^0) \xrightarrow{\quad \mathsf{d} \quad} \mathsf{S}(A^0) \otimes A^0 \xrightarrow{\quad 1\otimes\partial \quad} \mathsf{S}(A^0) \otimes A^1$$

# Proof Sketch

## Proof.

Let's go from an $\overline{S}$-algebra to a differential graded S-algebra.

# Proof Sketch

## Proof.

Let's go from an $\overline{\mathsf{S}}$-algebra to a differential graded S-algebra.

An $\overline{\mathsf{S}}$-algebra would be $\nu^\bullet : \overline{\mathsf{S}}\,(A^\bullet)^\bullet \to A^\bullet$.

# Proof Sketch

## Proof.

Let's go from an $\overline{\mathsf{S}}$-algebra to a differential graded S-algebra.

An $\overline{\mathsf{S}}$-algebra would be $\nu^\bullet : \overline{\mathsf{S}}\,(A^\bullet)^\bullet \to A^\bullet$.

Now since $\overline{\mathsf{S}}$ is an algebra modality, we know that $\overline{\mathsf{S}}$-algebras are commutative monoids, so in particular differential graded algebras. So $A^\bullet$ is a differential graded algebra, so we only really need to explain why $A^0$ is an S-algebra and $\partial : A^0 \to A^1$ is an S-derivation.

# Proof Sketch

## Proof.

Let's go from an $\overline{\mathsf{S}}$-algebra to a differential graded S-algebra.

An $\overline{\mathsf{S}}$-algebra would be $\nu^\bullet : \overline{\mathsf{S}}\,(A^\bullet)^\bullet \to A^\bullet$.

Now since $\overline{\mathsf{S}}$ is an algebra modality, we know that $\overline{\mathsf{S}}$-algebras are commutative monoids, so in particular differential graded algebras. So $A^\bullet$ is a differential graded algebra, so we only really need to explain why $A^0$ is an S-algebra and $\partial : A^0 \to A^1$ is an S-derivation.

Now from our chain complex morphism we get $\nu^0 : \overline{\mathsf{S}}\,(A^\bullet)^0 \to A^0$, or in other words, a map $\nu^0 : \mathsf{S}(A^0) \to A^0$. This indeed makes $A^0$ into a S-algebra thanks to our lifting result.

# Proof Sketch

## Proof.

Let's go from an $\overline{\mathsf{S}}$-algebra to a differential graded S-algebra.

An $\overline{\mathsf{S}}$-algebra would be $\nu^\bullet : \overline{\mathsf{S}}\,(A^\bullet)^\bullet \to A^\bullet$.

Now since $\overline{\mathsf{S}}$ is an algebra modality, we know that $\overline{\mathsf{S}}$-algebras are commutative monoids, so in particular differential graded algebras. So $A^\bullet$ is a differential graded algebra, so we only really need to explain why $A^0$ is an S-algebra and $\partial : A^0 \to A^1$ is an S-derivation.

Now from our chain complex morphism we get $\nu^0 : \overline{\mathsf{S}}\,(A^\bullet)^0 \to A^0$, or in other words, a map $\nu^0 : \mathsf{S}(A^0) \to A^0$. This indeed makes $A^0$ into a S-algebra thanks to our lifting result.

We also have $n^1 : \overline{\mathsf{S}}\,(A^\bullet)^1 \to A^1$, or in other words, a map $\nu^1 : \mathsf{S}(A^0) \otimes A^1 \to A^1$ which we can show is precisely the composite:

$$\mathsf{S}(A^0) \otimes A^1 \xrightarrow{\nu^0 \otimes 1} A^0 \otimes A^1 \xrightarrow{\mathsf{m}_{0,1}} A^1$$

□

# Proof Sketch

## Proof.

Then asking that $\nu^\bullet$ is a chain complex morphism means that the following diagram commutes:

$$\begin{array}{ccccc}
\mathsf{S}(A^0) & \xrightarrow{\ \mathrm{d}\ } & \mathsf{S}(A^0) \otimes A^0 & \xrightarrow{\ 1\otimes\partial\ } & \mathsf{S}(A^0) \otimes A^1 \\
\downarrow{\scriptstyle \nu^0} & & & & \downarrow{\scriptstyle \nu^1} \\
A^0 & & \xrightarrow[\ \partial\ ]{} & & A^1
\end{array}$$

which by the previous slide is precisely saying that $\partial : A^0 \to A^1$ is an S-derivation.

Going from a differential graded S-algebra to a $\overline{\mathsf{S}}$-algebra essentially does the reverse process. □

# Side Note: S-derivations are also $\overline{\mathsf{S}}$-algebras

- If you cut chain complexes to just have the degrees 0 and 1, you just gets maps of your category and therefore the arrow category $\mathsf{Arr}[\mathbb{X}]$

## Theorem

*Let $\mathbb{X}$ be a differential category, with differential modality $\mathsf{S}$, and also finite biproducts*[a]*.*

*Then $\mathsf{Arr}(\mathbb{X})$ is a differential category with differential modality $\overline{\mathsf{S}}$ defined as in the previous slide.*

*Moreover, $\overline{\mathsf{S}}$-algebras correspond precisely to $\mathsf{S}$-derivations.*

[a] We don't need negatives or symmetrized/exterior powers

# de Rham Complex is a Free Differential Graded S-Algebra

In his thesis, K. O'Neill

O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)

constructed the notion of the de Rham complex $\Omega^{\bullet}_{(A,nu)}$ for an S-algebra $(A, \nu)$ (if enough coequalizer exist).

If it exists, then the de Rham complex $\Omega^{\bullet}_{(A,nu)}$ is the free differential graded S-algebra over $(A, \nu)$.

# de Rham Complex is a Free Differential Graded S-Algebra

In his thesis, K. O'Neill

O'Neill, K. **Smoothness in codifferential categories** (PhD Thesis) (2017)

constructed the notion of the de Rham complex $\Omega^{\bullet}_{(A,nu)}$ for an S-algebra $(A, \nu)$ (if enough coequalizer exist).

If it exists, then the de Rham complex $\Omega^{\bullet}_{(A,nu)}$ is the free differential graded S-algebra over $(A, \nu)$.

Moreover, for free S-algebras $(\mathsf{S}(A), \mu)$, its de Rham complex can always be constructed (under the assumptions we have already given):

$$\Omega^{n}_{(\mathsf{S}(A),\mu)} = \mathsf{S}(A) \otimes \bigwedge^{n} A$$

Thus we always get a functor $\mathsf{KL}(\mathsf{S}) \to \mathsf{ALG}(\overline{\mathsf{S}})$.

With enough coequalizers, we get a functor $\mathsf{ALG}(\mathsf{S}) \to \mathsf{ALG}(\overline{\mathsf{S}})$.

# Towards differential dg-categories

$\mathsf{Ch}(\mathbb{X})$ is both a differential category and a dg-category.

It would be interesting to study the more general notion of a differential dg-category.

This will lead us to Cartesian differential dg-categories and tangent dg-categories as well.

Which would then introduce homotopy theory into the world of differential categories.

# Main Takeaways

- Differential graded algebras in a differential category.
- The category of chain complexes of a differential category is a differential category
- Differential graded algebras are the algebras of the differential modality on chain complexes.
- Working our way towards combining diff. category, dg-categories, and homotopy theory.

# That's all folks! Hope you enjoyed!

If you find differential/tangent categories interesting and have ideas, I hope you will start working with them! I am always happy to chat about differential categories, so feel free to come to talk to me or reach out by email. Also if want to come visit our CT group at Macquarie: also come talk to me, we love having visitors.

# That's all folks! Hope you enjoyed!

If you find differential/tangent categories interesting and have ideas, I hope you will start working with them! I am always happy to chat about differential categories, so feel free to come to talk to me or reach out by email. Also if want to come visit our CT group at Macquarie: also come talk to me, we love having visitors.

*Prague*

*Vienna*

*Brno*

**Thanks for listening! Merci!**

Email: `js.lemay@mq.edu.au`

Website: `https://sites.google.com/view/jspl-personal-webpage`