Extensional concepts in intensional type theory, revisited

Krzysztof Kapulkin and Yufeng Li

References

Main result

Kapulkin, Krzysztof and Li, Yufeng. Extensional concepts in intensional type theory, revisited. Theoretical Computer Science, 2025.

Background

Hofmann, Martin. Extensional constructs in intensional type theory. PhD thesis, 1995.

Kapulkin, Krzysztof and Lumsdaine, Peter LeFanu. The homotopy theory of type theories. Advances in Mathematics, 2018.

Isaev, Valery. Morita equivalences between algebraic dependent type theories. arXiv:1804.05045, 2020.

Two Notions of Equalities

Definitional	Propositional
$\vdash a_1 = a_2 : A$	$\vdash p : \mathsf{Id}_A(a_1, a_2)$

Dependent type theory with propositional equality gives intensional type theory (ITT).

Two Notions of Equalities

Definitional

$$\vdash a_1 = a_2 : A$$

Propositional

$$\vdash p : \mathsf{Id}_A(a_1, a_2)$$

Dependent type theory with propositional equality gives intensional type theory (ITT).

Equality reflection rule

Computation

$$\frac{\vdash a_1 : A \qquad \vdash a_2 : A \qquad \vdash p : \mathsf{Id}_A(a_1, a_2)}{\vdash a_1 = a_2 : A}$$

Adding equality reflection gives extensional type theory (ETT).

Two Notions of Equalities

Definitional	Propositional
$\vdash a_1 = a_2 : A$	$\vdash p : \mathrm{Id}_A(a_1, a_2)$

Dependent type theory with propositional equality gives intensional type theory (ITT).

Equality reflection rule

Computation	Logic
$\dfrac{\begin{array}{c}\vdash a_1 : A \qquad \vdash a_2 : A \\ \vdash p : \mathrm{Id}_A(a_1, a_2)\end{array}}{\vdash a_1 = a_2 : A}$	Provably equal $\|$ Seems reasonable $\Downarrow$ Definitionally equal

Adding equality reflection gives extensional type theory (ETT).

Two Notions of Equalities

Definitional	Propositional
$\vdash a_1 = a_2 : A$	$\vdash p : \mathsf{Id}_A(a_1, a_2)$

Dependent type theory with propositional equality gives intensional type theory (ITT).

Equality reflection rule

Computation	Logic	Topology
$$\frac{\vdash a_1 : A \qquad \vdash a_2 : A \qquad \vdash p : \mathsf{Id}_A(a_1, a_2)}{\vdash a_1 = a_2 : A}$$	Provably equal $\|$ Seems reasonable $\Downarrow$ Definitionally equal	Contractible $\|$ Not true in general $\Downarrow$ Singleton

Adding equality reflection gives extensional type theory (ETT).

Definitional vs. Propositional Equalities

Substitution vs. transport

Definitional	Propositional
$t = t'$	$p : \mathsf{Id}(t, t')$
$B(t) = B(t')$	$B(t) \xrightarrow{p_*} B(t')$

Definitional vs. Propositional Equalities

Substitution vs. transport

Definitional	Propositional
$t = t'$	$p : \mathsf{Id}(t, t')$
$B(t) = B(t')$	$B(t) \xrightarrow{p_*} B(t')$

- Changing terms between types indexed by definitionally equal terms is proof-independent.

Definitional vs. Propositional Equalities

Substitution vs. transport

Definitional	Propositional
$t = t'$	$p : \mathsf{Id}(t, t')$
$B(t) = B(t')$	$B(t) \xrightarrow{p_*} B(t')$

- Changing terms between types indexed by definitionally equal terms is proof-independent.
- Changing terms between types indexed by propositionally equal terms depends on the proof of equality.

Uniqueness of Identity Proofs (UIP)

$$\frac{\vdash p, p' : \mathsf{Id}_A(a_1, a_2)}{\vdash \mathsf{UIP}(p, p') : \mathsf{Id}(p, p')}$$

Uniqueness of identity proofs

Homotopically discrete space

Uniqueness of Identity Proofs (UIP)

$$\frac{\vdash p, p' : \mathsf{Id}_A(a_1, a_2)}{\vdash \mathsf{UIP}(p, p') : \mathsf{Id}(p, p')}$$

Uniqueness of identity proofs

Homotopically discrete space

Theorem (Hofmann 1995)

ETT is conservative over ITT+UIP.

$$\frac{\vdash p, p' : \mathsf{Id}_A(a_1, a_2)}{\vdash \mathsf{UIP}(p, p') : \mathsf{Id}(p, p')} \quad \leftrightsquigarrow \quad \frac{\vdash p : \mathsf{Id}_A(a_1, a_2)}{\vdash a_1 = a_2 : A}$$

Uniqueness of Identity Proofs (UIP)

$\dfrac{\vdash p, p' : \mathrm{Id}_A(a_1, a_2)}{\vdash \mathrm{UIP}(p, p') : \mathrm{Id}(p, p')}$	Uniqueness of identity proofs	Homotopically discrete space

Theorem (Hofmann 1995)

ETT is conservative over ITT+UIP.

$$\frac{\vdash p, p' : \mathrm{Id}_A(a_1, a_2)}{\vdash \mathrm{UIP}(p, p') : \mathrm{Id}(p, p')} \quad \leftrightsquigarrow \quad \frac{\vdash p : \mathrm{Id}_A(a_1, a_2)}{\vdash a_1 = a_2 : A}$$

Limitation. Syntactic result did not account for extensions.

Morita Equivalence of Type Theories

Definition

Two rings R and S are Morita equivalent iff $\mathsf{Mod}_R \simeq \mathsf{Mod}_S$.

Morita Equivalence of Type Theories

Definition

Two rings R and S are Morita equivalent iff $\mathsf{Mod}_R \simeq \mathsf{Mod}_S$.

$$\begin{pmatrix}\text{Equivalence of} \\ \text{type theories}\end{pmatrix} \overset{\text{def}}{=} \text{Morita equivalence}$$

Morita Equivalence of Type Theories

Definition

Two rings R and S are Morita equivalent iff $\mathrm{Mod}_R \simeq \mathrm{Mod}_S$.

$$\begin{pmatrix}\text{Equivalence of}\\ \text{type theories}\end{pmatrix} \overset{\text{def}}{=} \text{Morita equivalence} \overset{\text{def}}{=} \begin{pmatrix}\text{Equivalence between}\\ \text{categories of models}\end{pmatrix}$$

Morita Equivalence of Type Theories

Definition

Two rings R and S are Morita equivalent iff $\mathrm{Mod}_R \simeq \mathrm{Mod}_S$.

$$\begin{pmatrix}\text{Equivalence of}\\ \text{type theories}\end{pmatrix} \overset{\text{def}}{=} \text{Morita equivalence} \overset{\text{def}}{=} \begin{pmatrix}\text{Equivalence between}\\ \text{categories of models}\end{pmatrix}$$

Need to Determine

1. What is a model of a type theory?
2. What is a suitable notion of equivalence between categories of models?

Contextual Categories

Definition

A contextual category (C-system) structure on a category $\mathbb{C}$ consists of

Contextual Categories

Definition

A contextual category (C-system) structure on a category $\mathbb{C}$ consists of

Grading

$$\operatorname{ob}\mathbb{C} = \coprod_{n\in\mathbb{N}} \operatorname{ob}_n \mathbb{C}$$

Contextual Categories

Definition

A contextual category (C-system) structure on a category $\mathbb{C}$ consists of

Grading	Truncation
$\operatorname{ob}\mathbb{C} = \coprod_{n\in\mathbb{N}} \operatorname{ob}_n \mathbb{C}$	$\operatorname{ob}_{n+1}\mathbb{C} \xrightarrow{\text{ft}} \operatorname{ob}_n \mathbb{C}$

Notation. If $\operatorname{ft} A = \Gamma$ we write $A = \Gamma.A$.

Contextual Categories

Definition

A contextual category (C-system) structure on a category $\mathbb{C}$ consists of

Grading	Truncation	Projection
$\operatorname{ob}\mathbb{C} = \coprod_{n\in\mathbb{N}} \operatorname{ob}_n \mathbb{C}$	$\operatorname{ob}_{n+1}\mathbb{C} \xrightarrow{\operatorname{ft}} \operatorname{ob}_n \mathbb{C}$	$\Gamma.A \xrightarrow{\pi} \Gamma$

Notation. If $\operatorname{ft} A = \Gamma$ we write $A = \Gamma.A$.

Contextual Categories

Definition

A contextual category (C-system) structure on a category $\mathbb{C}$ consists of

Grading	Truncation	Projection
$\mathrm{ob}\,\mathbb{C} = \coprod_{n\in\mathbb{N}} \mathrm{ob}_n\,\mathbb{C}$	$\mathrm{ob}_{n+1}\,\mathbb{C} \xrightarrow{\mathrm{ft}} \mathrm{ob}_n\,\mathbb{C}$	$\Gamma.A \xrightarrow{\pi} \Gamma$

Notation. If $\mathrm{ft}\,A = \Gamma$ we write $A = \Gamma.A$.

Substitutions

$$\begin{tikzcd}
\Delta.f^*A \arrow[r, "f.A"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "\pi"] \\
\Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd}$$

Homotopies in Contextual Categories

$$\frac{\vdash A \text{ Type}}{(x_1, x_2 : A) \vdash \mathrm{Id}_A(x_1, x_2) \text{ Type}}$$

Path object

Provable equality

Definition

A homotopy $H\colon f \sim g$ between $f, g\colon \Gamma \to \Delta \in \mathbb{C}$

Homotopies in Contextual Categories

$$\frac{\vdash A \text{ Type}}{(x_1, x_2 : A) \vdash \mathrm{Id}_A(x_1, x_2) \text{ Type}}$$

Path object

Provable equality

Definition

A homotopy $H\colon f \sim g$ between $f, g\colon \Gamma \to \Delta \in \mathbb{C}$ is a factorisation

$$\begin{tikzcd}
\Gamma \arrow[rr, "{(f,g)}"] \arrow[dr, "H"'] & & \Delta \times \Delta \\
& \Delta.\Delta.\mathrm{Id}_\Delta \arrow[ur, two heads] &
\end{tikzcd}$$

Homotopies in Contextual Categories

$$\frac{\vdash A \text{ Type}}{(x_1, x_2 : A) \vdash \mathsf{Id}_A(x_1, x_2)\ \text{Type}}$$

Path object

Provable equality

Definition

A homotopy $H\colon f \sim g$ between $f, g\colon \Gamma \to \Delta \in \mathbb{C}$ is a factorisation

$$\begin{tikzcd}
\Gamma \arrow[rr, "{(f,g)}"] \arrow[dr, "H"'] & & \Delta \times \Delta \\
& \Delta.\Delta.\mathsf{Id}_\Delta \arrow[ur, two heads] &
\end{tikzcd}$$

Homotopy equivalences $w\colon \Gamma \to \Delta$ are those maps admitting left and right homotopy inverses.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Proposition (Weak Head Normalisation Property of Types)

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Proposition (Weak Head Normalisation Property of Types)

Suppose $\mathbb{C}$ is cellular.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Proposition (Weak Head Normalisation Property of Types)

Suppose $\mathbb{C}$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb{C}, T_i \in \mathsf{Ty}_{\mathbb{C}}\Theta_i)\}_{i \in I}$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Proposition (Weak Head Normalisation Property of Types)

Suppose $\mathbb{C}$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb{C}, T_i \in \mathsf{Ty}_{\mathbb{C}}\Theta_i)\}_{i \in I}$ such that if $X \in \mathsf{Ty}_{\mathbb{C}}\Gamma$ then precisely one of the following cases are true:

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Proposition (Weak Head Normalisation Property of Types)

Suppose $\mathbb{C}$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb{C}, T_i \in \mathsf{Ty}_{\mathbb{C}}\Theta_i)\}_{i\in I}$ such that if $X \in \mathsf{Ty}_{\mathbb{C}}\Gamma$ then precisely one of the following cases are true:

- Σ-type. $X = \Sigma(A, B)$ for some $A.B \in \mathsf{Ty}_{\mathbb{C}}\Gamma$.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Proposition (Weak Head Normalisation Property of Types)

Suppose $\mathbb{C}$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb{C}, T_i \in \mathsf{Ty}_{\mathbb{C}}\Theta_i)\}_{i\in I}$ such that if $X \in \mathsf{Ty}_{\mathbb{C}}\Gamma$ then precisely one of the following cases are true:

- Σ-type. $X = \Sigma(A, B)$ for some $A.B \in \mathsf{Ty}_{\mathbb{C}}\Gamma$.
- Π-type. $X = \Pi(A, B)$ for some $A.B \in \mathsf{Ty}_{\mathbb{C}}\Gamma$.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Proposition (Weak Head Normalisation Property of Types)

Suppose $\mathbb{C}$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb{C}, T_i \in \mathsf{Ty}_{\mathbb{C}}\Theta_i)\}_{i \in I}$ such that if $X \in \mathsf{Ty}_{\mathbb{C}}\Gamma$ then precisely one of the following cases are true:

- Σ-type. $X = \Sigma(A, B)$ for some $A.B \in \mathsf{Ty}_{\mathbb{C}}\Gamma$.
- Π-type. $X = \Pi(A, B)$ for some $A.B \in \mathsf{Ty}_{\mathbb{C}}\Gamma$.
- Id-type. $X = f^*\mathsf{Id}_A$ for some $f : \Gamma \to \Delta.A.A$.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.

Proposition (Weak Head Normalisation Property of Types)

Suppose $\mathbb{C}$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb{C}, T_i \in \mathsf{Ty}_{\mathbb{C}}\Theta_i)\}_{i \in I}$ such that if $X \in \mathsf{Ty}_{\mathbb{C}}\Gamma$ then precisely one of the following cases are true:

- Σ-type. $X = \Sigma(A, B)$ for some $A.B \in \mathsf{Ty}_{\mathbb{C}}\Gamma$.
- Π-type. $X = \Pi(A, B)$ for some $A.B \in \mathsf{Ty}_{\mathbb{C}}\Gamma$.
- Id-type. $X = f^*\mathsf{Id}_A$ for some $f : \Gamma \to \Delta.A.A$.
- Base type. $X = f^*T_i$ some unique (Θ_i, T_i) and $f : \Gamma \to \Theta_i$.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] \\
\mathbb{C}
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] \\
\mathbb{C}
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] & & \\
\mathbb{C} & & A
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] & \overline{A} & \\
\mathbb{C} & & A
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

$$\begin{tikzcd} \mathbb{D} \arrow[d, "F"] & \overline{A} \arrow[d, mapsto] & \\ \mathbb{C} & F\overline{A} & A \end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] \\
\mathbb{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\overline{A} \arrow[d, mapsto] & \\
F\overline{A} \arrow[r, dashed, "\cong"] & A
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

Weak term lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] & \overline{A} \arrow[d, mapsto] & \\
\mathbb{C} & F\overline{A} \arrow[r, dashed, "\simeq"] & A
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

Weak term lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] \\
\mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{A} \arrow[d, mapsto] & \\
F\overline{A} \arrow[r, dashed, "\simeq"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{A} & \overline{B}
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] \\
\mathbb{C}
\end{tikzcd}$$

Weak type lifting

$$\begin{tikzcd}
\overline{A} \arrow[d, mapsto] & \\
F\overline{A} \arrow[r, dashed, "\simeq"] & A
\end{tikzcd}$$

Weak term lifting

$$\begin{tikzcd}
\overline{A} \arrow[d, mapsto] & \overline{B} \arrow[d, mapsto] \\
F\overline{A} & F\overline{B}
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] \\
\mathbb{C}
\end{tikzcd}$$

Weak type lifting

$$\begin{tikzcd}
\overline{A} \arrow[d, mapsto] & \\
F\overline{A} \arrow[r, dashed, "\simeq"] & A
\end{tikzcd}$$

Weak term lifting

$$\begin{tikzcd}
\overline{A} \arrow[d, mapsto] & \overline{B} \arrow[d, mapsto] \\
F\overline{A} \arrow[r, bend right, "t"'] & F\overline{B}
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathrm{CxlCat}_{\mathrm{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

Weak term lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] \\
\mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{A} \arrow[d, mapsto] & \\
F\overline{A} \arrow[r, dashed, "\simeq"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{A} \arrow[r, dashed, "\overline{t}"] \arrow[d, mapsto] & \overline{B} \arrow[d, mapsto] \\
F\overline{A} \arrow[r, bend right, "t"'] & F\overline{B}
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] & \overline{A} \arrow[d, maps to] & \\
\mathbb{C} & F\overline{A} \arrow[r, dashed, "\simeq"] & A
\end{tikzcd}$$

Weak term lifting

$$\begin{tikzcd}
\overline{A} \arrow[rr, dashed, "\overline{t}"] \arrow[d, maps to] & {} \arrow[d, maps to] & \overline{B} \arrow[d, maps to] \\
F\overline{A} \arrow[rr, bend left, "F\overline{t}"] \arrow[rr, bend right, "t"'] & {} & F\overline{B}
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Weak type lifting

$$\begin{tikzcd}
\mathbb{D} \arrow[d, "F"] \\
\mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{A} \arrow[d, mapsto] & \\
F\overline{A} \arrow[r, dashed, "\simeq"] & A
\end{tikzcd}$$

Weak term lifting

$$\begin{tikzcd}
\overline{A} \arrow[rr, dashed, "\overline{t}"] \arrow[d, mapsto] & {} \arrow[d, mapsto] & \overline{B} \arrow[d, mapsto] \\
F\overline{A} \arrow[rr, bend left, "F\overline{t}"] \arrow[rr, bend right, "t"'] \arrow[rr, phantom, "\sim"] & {} & F\overline{B}
\end{tikzcd}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

$$\begin{pmatrix}\text{Equivalence of}\\ \text{type theories}\end{pmatrix} \overset{\text{def}}{=} \text{Morita equivalence} \overset{\text{def}}{=} \begin{pmatrix}\text{Equivalence between}\\ \text{categories of models}\end{pmatrix}$$

Morita Equivalence

Theorem (Kapulkin–Lumsdaine 2018)

The category $\mathsf{CxlCat}_{\mathsf{ITT}}$ of models of ITT admits a cofibrantly-generated left semi-model structure.

- Relative cell complexes are syntactic extensions.
- Weak equivalences are maps whose total right derived functor is an equivalence of categories.

$$\left(\begin{array}{c}\text{Equivalence of}\\ \text{type theories}\end{array}\right) \overset{\text{def}}{=} \text{Morita equivalence} \overset{\text{def}}{=} \left(\begin{array}{c}\text{Equivalence between}\\ \text{categories of models}\end{array}\right)$$

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

$$\begin{tikzcd}\mathsf{CxlCat}_{\mathbb{T}_1} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathbb{T}_2} \arrow[l, shift left, "U"]\end{tikzcd}$$

Connection with Logical Power

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

Connection with Logical Power

Quillen equivalence says the adjunction unit $\mathbb{C} \to UF\mathbb{C}$ at cell complexes is a weak equivalence.

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

Connection with Logical Power

Quillen equivalence says the adjunction unit $\mathbb{C} \to UF\mathbb{C}$ at cell complexes is a weak equivalence.

- If $\mathbb{C}$ is a model of $\mathbb{T}_1$ extended with base types, terms and propositional equalities

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

$$\begin{tikzcd}\mathsf{CxlCat}_{\mathbb{T}_1} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathbb{T}_2} \arrow[l, shift left, "U"]\end{tikzcd}$$

Connection with Logical Power

Quillen equivalence says the adjunction unit $\mathbb{C} \to UF\mathbb{C}$ at cell complexes is a weak equivalence.

- If $\mathbb{C}$ is a model of $\mathbb{T}_1$ extended with base types, terms and propositional equalities
- ...then there is an associated model $F\mathbb{C}$ of $\mathbb{T}_2$

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

Connection with Logical Power

Quillen equivalence says the adjunction unit $\mathbb{C} \to UF\mathbb{C}$ at cell complexes is a weak equivalence.

- If $\mathbb{C}$ is a model of $\mathbb{T}_1$ extended with base types, terms and propositional equalities
- ...then there is an associated model $F\mathbb{C}$ of $\mathbb{T}_2$
- ...such that if we compile back to $UF\mathbb{C}$ as a model of $\mathbb{T}_1$

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

$$\begin{tikzcd}\mathsf{CxlCat}_{\mathbb{T}_1} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathbb{T}_2} \arrow[l, shift left, "U"]\end{tikzcd}$$

Connection with Logical Power

Quillen equivalence says the adjunction unit $\mathbb{C} \to UF\mathbb{C}$ at cell complexes is a weak equivalence.

- If $\mathbb{C}$ is a model of $\mathbb{T}_1$ extended with base types, terms and propositional equalities
- ...then there is an associated model $F\mathbb{C}$ of $\mathbb{T}_2$
- ...such that if we compile back to $UF\mathbb{C}$ as a model of $\mathbb{T}_1$
- ...then the expressible and provable statements in those two models are correspond propositionally within type theory.

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

Theorem (Isaev 2020)

Definitional and propositional equalities give rise to Morita equivalent formulations of the singleton type.

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

Theorem (Isaev 2020)

Definitional and propositional equalities give rise to Morita equivalent formulations of the singleton type.

- Propositional singleton: ITT+Contr

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

Theorem (Isaev 2020)

Definitional and propositional equalities give rise to Morita equivalent formulations of the singleton type.

- Propositional singleton: ITT+Contr
- Definitional singleton: ITT+Unit

Morita Equivalence

Definition (Morita Equivalence)

Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{CxlCat}_{\mathbb{T}_2}$.

Theorem (Isaev 2020)

Definitional and propositional equalities give rise to Morita equivalent formulations of the singleton type.

- Propositional singleton: ITT+Contr
- Definitional singleton: ITT+Unit

Removing Singleton Restriction

- Theory of propositional equalities: ITT
- Theory of definitional equalities: ETT

Conservativity of ITT+UIP over ETT, Revisited

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd}
\mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, shift left, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, hook, shift left, "|-|"]
\end{tikzcd}$$

Proof.

Conservativity of ITT+UIP over ETT, Revisited

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd} \mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, bend left, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, hook, bend left, "|-|"] \end{tikzcd}$$

Proof. All models of ETT also are also models of $\mathsf{ITT} + \mathsf{UIP}$, so there is an inclusion $|-|\colon \mathsf{CxlCat}_{\mathsf{ETT}} \hookrightarrow \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$.

Conservativity of ITT+UIP over ETT, Revisited

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd} \mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, shift left, hook, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, shift left, hook, "|-|"] \end{tikzcd}$$

Proof. All models of ETT also are also models of $\mathsf{ITT} + \mathsf{UIP}$, so there is an inclusion $|-|\colon \mathsf{CxlCat}_{\mathsf{ETT}} \hookrightarrow \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$. By cocompleteness, it has a left adjoint $\langle - \rangle$.

Conservativity of ITT+UIP over ETT, Revisited

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd}\mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, bend left, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, bend left, "|-|"]\end{tikzcd}$$

Proof. All models of ETT also are also models of ITT + UIP, so there is an inclusion $|-|\colon \mathsf{CxlCat}_{\mathsf{ETT}} \hookrightarrow \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$. By cocompleteness, it has a left adjoint $\langle - \rangle$.

It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class.

Conservativity of ITT+UIP over ETT, Revisited

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd} \mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, shift left, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, shift left, hook, "|-|"] \end{tikzcd}$$

Proof. All models of ETT also are also models of $\mathsf{ITT} + \mathsf{UIP}$, so there is an inclusion $|-| \colon \mathsf{CxlCat}_{\mathsf{ETT}} \hookrightarrow \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$. By cocompleteness, it has a left adjoint $\langle - \rangle$.

It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class. The cells are “syntactic”:

Conservativity of ITT+UIP over ETT, Revisited

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd} \mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, shift left, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, hook', shift left, "|-|"] \end{tikzcd}$$

Proof. All models of ETT also are also models of $\mathsf{ITT} + \mathsf{UIP}$, so there is an inclusion $|-| \colon \mathsf{CxlCat}_{\mathsf{ETT}} \hookrightarrow \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$. By cocompleteness, it has a left adjoint $\langle - \rangle$.

It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class. The cells are “syntactic”: obtained by freely adding types and terms but no definitional equalities.

Conservativity of ITT+UIP over ETT, Revisited

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd} \mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, bend left, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, bend left, hook, "|-|"] \end{tikzcd}$$

Proof. All models of ETT also are also models of ITT + UIP, so there is an inclusion $|-|\colon \mathsf{CxlCat}_{\mathsf{ETT}} \hookrightarrow \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$. By cocompleteness, it has a left adjoint $\langle - \rangle$.

It suffices to check $\mathbb{C} \to |\langle \mathbb{C} \rangle|$ is a weak equivalence when $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ is a cell-complex of the generating left class. The cells are "syntactic": obtained by freely adding types and terms but no definitional equalities. This makes it tractable to explicitly construct $\langle \mathbb{C} \rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$. ♡

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

A quotient construction

- To support equality reflection: must identify homotopic maps.

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

A quotient construction

- To support equality reflection: must identify homotopic maps.
- Cannot take Ho $\mathbb{C}$.

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

A quotient construction

- To support equality reflection: must identify homotopic maps.
- Cannot take $\mathrm{Ho}\ \mathbb{C}$.
    - $\mathrm{Ho}\ \mathbb{C}$ formally inverts homotopy equivalences.

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

A quotient construction

- To support equality reflection: must identify homotopic maps.
- Cannot take Ho $\mathbb{C}$.
  - Ho $\mathbb{C}$ formally inverts homotopy equivalences.
  - This collapses too much.

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

A quotient construction

- To support equality reflection: must identify homotopic maps.
- Cannot take Ho $\mathbb{C}$.
  - Ho $\mathbb{C}$ formally inverts homotopy equivalences.
  - This collapses too much.

Example. The map Bool → Bool swapping true and false is a propositional isomorphism but is not the identity even under equality reflection.

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

A quotient construction

- To support equality reflection: must identify homotopic maps.
- Cannot take Ho $\mathbb{C}$.
  - Ho $\mathbb{C}$ formally inverts homotopy equivalences.
  - This collapses too much.

  Example. The map Bool → Bool swapping true and false is a propositional isomorphism but is not the identity even under equality reflection.
- Upshot. $\langle\mathbb{C}\rangle$ is obtained from $\mathbb{C}$ by carefully choosing a wide subcategory $\mathcal{W}_{\mathsf{ETT}}$ of homotopy equivalences to collapse.

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

Construction

$\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is the category with

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

Construction

$\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is the category with

- Objects $\mathrm{ob}\,\mathbb{C}/\equiv$, where

$$\Gamma \equiv \Gamma' \Leftrightarrow \text{Exists some } \Gamma \simeq \Gamma' \in \mathcal{W}_{\mathsf{ETT}}$$

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

Construction

$\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is the category with

- Objects $\operatorname{ob}\mathbb{C}/\equiv$, where

$$\Gamma \equiv \Gamma' \Leftrightarrow \text{Exists some } \Gamma \simeq \Gamma' \in \mathcal{W}_{\mathsf{ETT}}$$

- Maps $\operatorname{mor}\mathbb{C}/\equiv$, where

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \left(\exists \begin{matrix} \Gamma \simeq \Gamma', \\ \Delta \simeq \Delta' \end{matrix} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

Construction

$\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is the category with

- Objects $\operatorname{ob}\mathbb{C}/\equiv$, where

$$\Gamma \equiv \Gamma' \Leftrightarrow \text{Exists some } \Gamma \simeq \Gamma' \in \mathcal{W}_{\mathsf{ETT}}$$

- Maps $\operatorname{mor}\mathbb{C}/\equiv$, where

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \left(\exists \begin{matrix} \Gamma \simeq \Gamma', \\ \Delta \simeq \Delta' \end{matrix} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

By construction, $\langle\mathbb{C}\rangle$ is extensional.

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

Construction

$\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is the category with

- Objects $\operatorname{ob}\mathbb{C}/\!\equiv$, where

$$\Gamma \equiv \Gamma' \Leftrightarrow \text{Exists some } \Gamma \simeq \Gamma' \in \mathcal{W}_{\mathsf{ETT}}$$

- Maps $\operatorname{mor}\mathbb{C}/\!\equiv$, where

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \left(\exists \begin{smallmatrix} \Gamma \simeq \Gamma', \\ \Delta \simeq \Delta' \end{smallmatrix} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

By construction, $\langle\mathbb{C}\rangle$ is extensional. The quotient map $[-] \colon \mathbb{C} \to |\langle\mathbb{C}\rangle|$ has the weak lifting property for Morita equivalence.

From $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ to $\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$

Construction

$\langle\mathbb{C}\rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is the category with

- Objects $\operatorname{ob}\mathbb{C}/{\equiv}$, where

$$\Gamma \equiv \Gamma' \Leftrightarrow \text{Exists some } \Gamma \simeq \Gamma' \in \mathcal{W}_{\mathsf{ETT}}$$

- Maps $\operatorname{mor}\mathbb{C}/{\equiv}$, where

$$\left(\begin{matrix}\Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta'\end{matrix}\right) \Leftrightarrow \left(\exists\, \begin{matrix}\Gamma \simeq \Gamma', \\ \Delta \simeq \Delta'\end{matrix} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

By construction, $\langle\mathbb{C}\rangle$ is extensional. The quotient map $[-]\colon \mathbb{C} \to |\langle\mathbb{C}\rangle|$ has the weak lifting property for Morita equivalence.

Example (Hofmann 1995). If $\mathbb{S}$ is the syntactic model, $\langle\mathbb{S}\rangle = \mathbb{Q}$ as from Hofmann.

Role of the UIP Axiom

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \left(\exists \begin{smallmatrix} \Gamma \simeq \Gamma', \\ \Delta \simeq \Delta' \end{smallmatrix} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach.

Role of the UIP Axiom

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \left(\exists \begin{smallmatrix} \Gamma \simeq \Gamma', \\ \Delta \simeq \Delta' \end{smallmatrix} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach.

$$\begin{tikzcd} \Gamma \arrow[r, "f"] & \Delta_1 & \Delta_2 \arrow[r, "g"] & \Theta \end{tikzcd}$$

Role of the UIP Axiom

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \left(\exists^{\Gamma \simeq \Gamma',}_{\Delta \simeq \Delta'} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach.

$$\Gamma \xrightarrow{f} \Delta_1 \xrightarrow{\simeq} \Delta_2 \xrightarrow{g} \Theta$$

Role of the UIP Axiom

$$\left(\begin{array}{c}\Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta'\end{array}\right) \Leftrightarrow \left(\exists^{\Gamma \simeq \Gamma'}_{\Delta \simeq \Delta'}, \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd}\right)$$

Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach. Need to show $f \equiv f'$ and $g \equiv g'$ composable then $gf \equiv g'f'$.

$$\begin{tikzcd} \Gamma \arrow[r, "f"] & \Delta_1 \arrow[r, "\simeq"] & \Delta_2 \arrow[r, "g"] & \Theta \\ \Gamma' \arrow[r, "f'"'] & \Delta'_1 \arrow[r, "\simeq"'] & \Delta'_2 \arrow[r, "g'"'] & \Theta' \end{tikzcd}$$

Role of the UIP Axiom

$$\left(\begin{array}{c} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{array}\right) \Leftrightarrow \left(\exists^{\Gamma \simeq \Gamma',}_{\Delta \simeq \Delta'} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach. Need to show $f \equiv f'$ and $g \equiv g'$ composable then $gf \equiv g'f'$.

$$\begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta_1 \arrow[r, "\simeq"] \arrow[d, dashed, "\simeq"] & \Delta_2 \arrow[r, "g"] \arrow[d, dashed, "\simeq"] \arrow[dr, phantom, "\sim"] & \Theta \arrow[d, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta_1' \arrow[r, "\simeq"'] & \Delta_2' \arrow[r, "g'"'] & \Theta' \end{tikzcd}$$

Role of the UIP Axiom

$$\left(\begin{array}{c} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{array}\right) \Leftrightarrow \left(\exists \begin{array}{c} \Gamma \simeq \Gamma', \\ \Delta \simeq \Delta' \end{array} \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta \arrow[d, dashed, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta' \end{tikzcd} \right)$$

Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach. Need to show $f \equiv f'$ and $g \equiv g'$ composable then $gf \equiv g'f'$. Amounts to showing the middle square commutes up to homotopy.

$$\begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\sim"] & \Delta_1 \arrow[r, "\simeq"] \arrow[d, dashed, "\simeq"] \arrow[dr, phantom, "\overset{?}{\sim}"] & \Delta_2 \arrow[r, "g"] \arrow[d, dashed, "\simeq"] \arrow[dr, phantom, "\sim"] & \Theta \arrow[d, "\simeq"] \\ \Gamma' \arrow[r, "f'"'] & \Delta_1' \arrow[r, "\simeq"'] & \Delta_2' \arrow[r, "g'"'] & \Theta' \end{tikzcd}$$

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Role of the UIP Axiom

Lemma

By UIP, if $w, w' \colon X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
& \Delta_2.f_2^*A_2 \arrow[rr] \arrow[dd, two heads] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^*A_1 \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[dd, two heads] & \\
& \Delta_2 \arrow[rr, "f_2"] & & \Gamma_2 \\
\Delta_1 \arrow[rr, "f_1"'] & & \Gamma_1 &
\end{tikzcd}$$

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
 & \Delta_2.f_2^* A_2 \arrow[rr] \arrow[dd, two heads] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^* A_1 \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ur, dashed, "\simeq"] \arrow[dd, two heads] & \\
 & \Delta_2 \arrow[rr, "f_2"] & & \Gamma_2 \\
\Delta_1 \arrow[rr, "f_1"'] \arrow[ur, dashed, "\simeq"] & & \Gamma_1 \arrow[ur, dashed, "\simeq"] &
\end{tikzcd}$$

If the solid maps above are in $\mathcal{W}_{\mathsf{ETT}}$

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
 & \Delta_2.f_2^* A_2 \arrow[rr] \arrow[dd, two heads] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^* A_1 \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ur, dashed, "\simeq"] \arrow[dd, two heads] & \\
 & \Delta_2 \arrow[rr, "f_2"] & & \Gamma_2 \\
\Delta_1 \arrow[ur, dashed, "\simeq"] \arrow[rr, "f_1"'] \arrow[urrr, phantom, "\sim"] & & \Gamma_1 \arrow[ur, dashed, "\simeq"] &
\end{tikzcd}$$

If the solid maps above are in $\mathcal{W}_{\mathsf{ETT}}$ and bottom face commute up to homotopy

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
 & \Delta_2.f_2^*A_2 \arrow[rr] \arrow[dd, two heads, "\sim"] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^*A_1 \arrow[ur, dashed, "\simeq"] \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ur, dashed, "\simeq"] \arrow[dd, two heads] & \\
 & \Delta_2 \arrow[rr, "f_2"] \arrow[dr, phantom, "\sim"] & & \Gamma_2 \\
\Delta_1 \arrow[ur, dashed, "\simeq"] \arrow[rr, "f_1"'] & & \Gamma_1 \arrow[ur, dashed, "\simeq"] &
\end{tikzcd}$$

If the solid maps above are in $\mathcal{W}_{\mathsf{ETT}}$ and bottom face commute up to homotopy then induced map is in $\mathcal{W}_{\mathsf{ETT}}$.

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
& \Delta_2.f_2^* A_2 \arrow[rr] \arrow[dd, two heads] \arrow[dr, phantom, "\sim"] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^* A_1 \arrow[ur, dashed, "\simeq"] \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ur, dashed, "\simeq"] \arrow[dd, two heads] & \\
& \Delta_2 \arrow[rr, "f_2"] \arrow[dr, phantom, "\sim"] & & \Gamma_2 \\
\Delta_1 \arrow[ur, dashed, "\simeq"] \arrow[rr, "f_1"'] & & \Gamma_1 \arrow[ur, dashed, "\simeq"] &
\end{tikzcd}$$

Inductively: parallel purple maps are homotopic.

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
 & \Delta_2.f_2^* A_2 \arrow[rr] \arrow[dd, two heads, "\sim"] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^* A_1 \arrow[ur, dashed, "\simeq"] \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ur, dashed, "\simeq"] \arrow[dd, two heads] & \\
 & \Delta_2 \arrow[rr, "f_2"] \arrow[dr, phantom, "\sim"] & & \Gamma_2 \\
\Delta_1 \arrow[ur, dashed, "\simeq"] \arrow[rr, "f_1"'] & & \Gamma_1 \arrow[ur, dashed, "\simeq"] &
\end{tikzcd}$$

Inductively: parallel purple maps are homotopic. Show parallel orange maps are homotopic.

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
& \Delta_2.f_2^*A_2 \arrow[rr] \arrow[dd, two heads, "\sim"] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^*A_1 \arrow[ru, dashed, "\simeq"] \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ru, dashed, "\simeq"] \arrow[dd, two heads] & \\
& \Delta_2 \arrow[rr, "f_2", "\sim"'] & & \Gamma_2 \\
\Delta_1 \arrow[ru, "\simeq"] \arrow[rr, "f_1"'] & & \Gamma_1 \arrow[ru, dashed, "\simeq"] &
\end{tikzcd}$$

Inductively: parallel purple maps are homotopic. Show parallel orange maps are homotopic. Construction is homotopy invariant.

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
& \Delta_2.f_2^* A_2 \arrow[rr] \arrow[dd, two heads] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^* A_1 \arrow[ur, dashed, "\simeq"] \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ur, dashed, "\simeq"] \arrow[dd, two heads] \arrow[from=1-2, phantom, "\sim"] & \\
& \Delta_2 \arrow[rr, "f_2"] & & \Gamma_2 \\
\Delta_1 \arrow[ur, "\simeq"] \arrow[rr, "f_1"'] & & \Gamma_1 \arrow[ur, dashed, "\simeq"] \arrow[from=3-2, phantom, "\sim"] &
\end{tikzcd}$$

Show that any two homotopies H, H' for the bottom face are homotopic.

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
& \Delta_2.f_2^* A_2 \arrow[rr] \arrow[dd, two heads] \arrow[dr, phantom, "\sim"] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^* A_1 \arrow[ur, dashed, "\simeq"] \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ur, dashed, "\simeq"] \arrow[dd, two heads] & \\
& \Delta_2 \arrow[rr, "f_2"] \arrow[dr, phantom, "\sim"] & & \Gamma_2 \\
\Delta_1 \arrow[ur, "\simeq"] \arrow[rr, "f_1"'] & & \Gamma_1 \arrow[ur, "\simeq"] &
\end{tikzcd}$$

$$\frac{\Gamma \vdash H, H' : \mathsf{Id}_A(a_1, a_2)}{\Gamma \vdash \mathsf{UIP}(H, H') : \mathsf{Id}_{\mathsf{Id}_A(a_1, a_2)}(H, H')}$$

Show that any two homotopies H, H' for the bottom face are homotopic. Homotopies are equality proofs.

Role of the UIP Axiom

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.

$$\begin{tikzcd}
& \Delta_2.f_2^* A_2 \arrow[rr] \arrow[dd, two heads, "\sim"] & & \Gamma_2.A_2 \arrow[dd, two heads] \\
\Delta_1.f_1^* A_1 \arrow[ur, dashed, "\simeq"] \arrow[rr] \arrow[dd, two heads] & & \Gamma_1.A_1 \arrow[ur, dashed, "\simeq"] \arrow[dd, two heads] & \\
& \Delta_2 \arrow[rr, "f_2"] \arrow[dr, phantom, "\sim"] & & \Gamma_2 \\
\Delta_1 \arrow[ur, dashed, "\simeq"] \arrow[rr, "f_1"'] & & \Gamma_1 \arrow[ur, dashed, "\simeq"] &
\end{tikzcd}$$

$$\frac{\Gamma \vdash H, H' : \mathsf{Id}_A(a_1, a_2)}{\Gamma \vdash \mathsf{UIP}(H, H') : \mathsf{Id}_{\mathsf{Id}_A(a_1, a_2)}(H, H')}$$

Show that any two homotopies H, H' for the bottom face are homotopic. Homotopies are equality proofs. Follows by UIP.

Summary

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Summary

Lemma

By UIP, if $w, w' \colon X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Lemma

If $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ cellular then the quotient category $\langle \mathbb{C} \rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is a category with well-defined composition.

Summary

Lemma

By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

Lemma

If $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT+UIP}}$ cellular then the quotient category $\langle \mathbb{C} \rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is a category with well-defined composition.

Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$\begin{tikzcd}
\mathsf{CxlCat}_{\mathsf{ITT+UIP}} \arrow[r, shift left=1.5ex, "\langle - \rangle"] \arrow[r, phantom, "\perp"] & \mathsf{CxlCat}_{\mathsf{ETT}} \arrow[l, hook, shift left=1.5ex, "|-|"]
\end{tikzcd}$$

Conclusion

Future directions

- Constructive proof of Hofmann's result.
- Encompassing internal universes.
- Further instances of Morita equivalence.

Conclusion

Future directions

- Constructive proof of Hofmann's result.
- Encompassing internal universes.
- Further instances of Morita equivalence.

Thank you!