Extensional concepts in intensional type theory, revisited

Krzysztof Kapulkin and Yufeng Li



Main result

Kapulkin, Krzysztof and Li, Yufeng. Extensional concepts in intensional type theory, revisited. Theoretical Computer Science, 2025.



Background

Hofmann, Martin. Extensional constructs in intensional type theory. PhD thesis, 1995.

Kapulkin, Krzysztof and Lumsdaine, Peter LeFanu. The homotopy theory of type theories. Advances in Mathematics,

Isaev, Valery. Morita equivalences between algebraic dependent type theories. arXiv:1804.05045, 2020.

Definitional Propositional
$$\vdash a_1 = a_2 : A$$
 $\vdash p : Id_A(a_1, a_2)$

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Computation

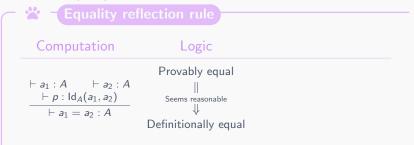
$$\vdash a_1 : A \qquad \vdash a_2 : A$$

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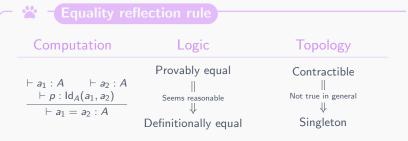
Adding equality reflection gives extensional type theory (ETT).

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Substitution vs. transport

Definitional	Propositional
t=t'	p: Id(t,t')
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Changing terms between types indexed by definitionally equal terms is proof-independent.



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- Changing terms between types indexed by definitionally equal terms is proof-independent.
- Changing terms between types indexed by propositionally equal terms depends on the proof of equality.

Uniqueness of Identity Proofs (UIP)

$$\frac{\vdash p, p' : \mathsf{Id}_{A}(a_1, a_2)}{\vdash \mathsf{UIP}(p, p') : \mathsf{Id}(p, p')} \quad \begin{array}{c} \mathsf{Uniqueness} \ \mathsf{of} \ \mathsf{identity} & \mathsf{Homotopically} \ \mathsf{discrete} \\ \mathsf{proofs} & \mathsf{space} \end{array}$$

$$\frac{\vdash p, p' : \mathsf{Id}_{\mathcal{A}}(a_1, a_2)}{\vdash \mathsf{UIP}(p, p') : \mathsf{Id}(p, p')} \quad \begin{array}{c} \mathsf{Uniqueness} \ \mathsf{of} \ \mathsf{identity} & \mathsf{Homotopically} \ \mathsf{discrete} \\ \mathsf{proofs} & \mathsf{space} \end{array}$$

Theorem (Hofmann 1995)

ETT is conservative over ITT+UIP.

$$\frac{\vdash p,p':\mathsf{Id}_A(a_1,a_2)}{\vdash \mathsf{UIP}(p,p'):\mathsf{Id}(p,p')} \iff \frac{\vdash p:\mathsf{Id}_A(a_1,a_2)}{\vdash a_1=a_2:A}$$

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Limitation. Syntactic result did not account for extensions.

Morita Equivalence of Type Theories





Two rings R and S are Morita equivalent iff $Mod_R \simeq Mod_S$.

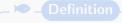


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$$\begin{pmatrix} \text{Equivalence of} \\ \text{type theories} \end{pmatrix} \stackrel{\text{def}}{=} \text{Morita equivalence}$$



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- 1. What is a model of a type theory?
- 2. What is a suitable notion of equivalence between categories of models?



A contextual category (C-system) structure on a category $\mathbb C$ consists of



Definition

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Grading

$$\mathsf{ob}\,\mathbb{C}=\coprod_{n\in\mathbb{N}}\mathsf{ob}_n\,\mathbb{C}$$



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Grading Truncation
$$\operatorname{ob} \mathbb{C} = \coprod_{n \in \mathbb{N}} \operatorname{ob}_n \mathbb{C} \qquad \operatorname{ob}_{n+1} \mathbb{C} \xrightarrow{\operatorname{ft}} \operatorname{ob}_n \mathbb{C}$$

Notation. If ft $A = \Gamma$ we write $A = \Gamma . A$.



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Grading Truncation Projection
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Substitutions

$$\begin{array}{ccc} \Delta.f^*A \xrightarrow{f.A} \Gamma.A \\ \downarrow^{\pi} & \downarrow^{\pi} \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

$$\frac{\vdash A \mathsf{Type}}{(x_1, x_2 : A) \vdash \mathsf{Id}_A(x_1, x_2) \mathsf{Type}}$$

Path object

Provable equality

A homotopy $H \colon f \sim g$ between $f,g \colon \Gamma \to \Delta \in \mathbb{C}$

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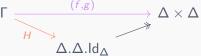
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Homotopy equivalences $w \colon \Gamma \to \Delta$ are those maps admitting left and right homotopy inverses.

Morita Equivalence



Theorem (Kapulkin-Lumsdaine 2018)

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Suppose $\mathbb C$ is cellular. Then, there is a set of base types $\{(\Theta_i \in \mathbb C, T_i \in \mathsf{Ty}_{\mathbb C}\Theta_i)\}_{i\in I}$ such that if $X \in \mathsf{Ty}_{\mathbb C}\Gamma$ then precisely one of the following cases are true:

▶ Σ-type. $X = \Sigma(A, B)$ for some $A.B \in \mathsf{Ty}_{\mathbb{C}}\Gamma$.

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- ▶ Id-type. $X = f^* Id_A$ for some $f : \Gamma \to \Delta.A.A$.
- ▶ Base type. $X = f^*T_i$ some unique (Θ_i, T_i) and $f: \Gamma \to \Theta_i$.

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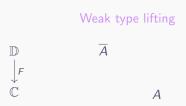
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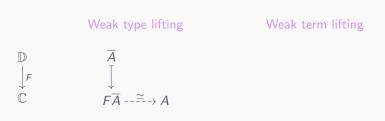






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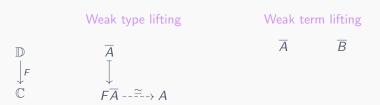
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Two type theories $\mathbb{T}_1, \mathbb{T}_2$ extending ITT are Morita equivalent if there is a Quillen equivalence $\mathsf{CxlCat}_{\mathbb{T}_1} \xrightarrow{\longleftarrow} \mathsf{CxlCat}_{\mathbb{T}_2}$.



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Quillen equivalence says the adjunction unit $\mathbb{C} \to UF\mathbb{C}$ at cell complexes is a weak equivalence.

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- ...then the expressible and provable statements in those two models are correspond propositionally within type theory.

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- Propositional singleton: ITT+Contr
- ► Definitional singleton: ITT+Unit
- Removing Singleton Restriction
- ► Theory of propositional equalities: ITT
- ► Theory of definitional equalities: ETT



The type theories ITT+UIP and ETT are Morita equivalent.

$$\mathsf{CxICat}_{\mathsf{ITT}+\mathsf{UIP}} \xleftarrow{\langle -\rangle}{\leftarrow \bot} \mathsf{CxICat}_{\mathsf{ETT}}$$



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$$CxICat_{ITT+UIP} \xrightarrow{\langle - \rangle} CxICat_{ETT}$$

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- ► To support equality reflection: must identify homotopic maps.



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From $\mathbb{C}\in\mathsf{CxlCat}_{\mathsf{ITT}+\mathsf{UIP}}$ to $\langle\mathbb{C} angle\in\mathsf{CxlCat}_{\mathsf{ETT}}$

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 - The map $Bool \rightarrow Bool$ swapping true and false is a propositional isomorphism but is not the identity even under equality reflection.
- ▶ Upshot. $\langle \mathbb{C} \rangle$ is obtained from \mathbb{C} by carefully choosing a wide subcategory $\mathcal{W}_{\mathsf{ETT}}$ of homotopy equivalences to collapse.

Construction

 $\langle \mathbb{C} \rangle \in \mathsf{CxICat}_{\mathsf{ETT}}$ is the category with

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 - ▶ Objects ob \mathbb{C}/\equiv , where

$$\Gamma \equiv \Gamma' \Leftrightarrow \mathsf{Exists} \mathsf{ some } \Gamma \simeq \Gamma' \in \mathcal{W}_{\mathsf{ETT}}$$

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▶ Maps mor \mathbb{C}/\equiv , where

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \begin{pmatrix} \exists \Gamma \simeq \Gamma', \\ \Delta \simeq \Delta' \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \xrightarrow{\simeq \downarrow} \sim \downarrow \simeq \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix}$$

$\mathsf{From}\ \mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT}+\mathsf{UIP}}\ \mathsf{to}\ \langle \mathbb{C} angle \in \mathsf{CxlCat}_{\mathsf{ET}}$

- Construction
- $\langle \mathbb{C} \rangle \in \mathsf{CxICat}_{\mathsf{ETT}}$ is the category with
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By construction, $\langle \mathbb{C} \rangle$ is extensional.

From $\mathbb{C}\in\mathsf{CxlCat}_{\mathsf{ITT}+\mathsf{UIP}}$ to $\langle\mathbb{C} angle\in\mathsf{CxlCat}_{\mathsf{ETT}}$

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By construction, $\langle \mathbb{C} \rangle$ is extensional. The quotient map $[-]: \mathbb{C} \to |\langle \mathbb{C} \rangle|$ has the weak lifting property for Morita equivalence.

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- $\langle \mathbb{C} \rangle \in \mathsf{CxICat}_{\mathsf{ETT}}$ is the category with
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By construction, $\langle \mathbb{C} \rangle$ is extensional. The quotient map $[-]: \mathbb{C} \to |\langle \mathbb{C} \rangle|$ has the weak lifting property for Morita equivalence.

T Example (Hofmann 1995). If $\mathbb S$ is the syntactic model, $\langle \mathbb S \rangle = \mathbb Q$ as from Hofmann.

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \begin{pmatrix} \exists_{\Delta \simeq \Delta'}^{\Gamma \simeq \Gamma'}, \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \xrightarrow{\simeq \downarrow} & \xrightarrow{\Gamma} & \xrightarrow{L} \\ \Gamma' \xrightarrow{f'} & \Delta' \end{pmatrix}$$

H _ Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach.

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \begin{pmatrix} \exists_{\Delta \simeq \Delta'}^{\Gamma \simeq \Gamma'}, \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \xrightarrow{\simeq_{+}^{1}} \xrightarrow{\sim} \xrightarrow{\downarrow \simeq} \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix}$$

Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach.

$$\Gamma \stackrel{f}{\longrightarrow} \Delta$$

$$\Gamma \stackrel{f}{\longrightarrow} \Delta_1 \qquad \quad \Delta_2 \stackrel{g}{\longrightarrow} \Theta$$

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \begin{pmatrix} \exists_{\Delta \simeq \Delta'}^{\Gamma \simeq \Gamma'}, \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \xrightarrow{\simeq \downarrow}^{\Gamma} \xrightarrow{f} \Delta \\ \vdash \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix}$$

☆ _ Lemma

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Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach. Need to show $f \equiv f'$ and $g \equiv g'$ composable then $gf \equiv g'f'$.

$$\Gamma \stackrel{\textit{f}}{\longrightarrow} \Delta_1 \stackrel{\simeq}{\longrightarrow} \Delta_2 \stackrel{\textit{g}}{\longrightarrow} \Theta$$

$$\Gamma' \xrightarrow[f']{f'} \Delta'_1 \xrightarrow{\simeq} \Delta'_2 \xrightarrow[g']{g'} \Theta'$$

$$\begin{pmatrix} \Gamma \xrightarrow{f} \Delta \\ \equiv \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix} \Leftrightarrow \begin{pmatrix} \exists_{\Delta \simeq \Delta'}^{\Gamma \simeq \Gamma'}, \in \mathcal{W}_{\mathsf{ETT}} \text{ st. } \xrightarrow{\simeq_{+}^{1}} \xrightarrow{\sim} \xrightarrow{\downarrow \simeq} \\ \Gamma' \xrightarrow{f'} \Delta' \end{pmatrix}$$

_ 🕌 _ Lemma

Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach. Need to show $f \equiv f'$ and $g \equiv g'$ composable then $gf \equiv g'f'$.

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Define composition in $\langle \mathbb{C} \rangle$ and show well-definedness.

Proof. Replicate Hofmann's approach. Need to show $f \equiv f'$ and $g \equiv g'$ composable then $gf \equiv g'f'$. Amounts to showing the middle square commutes up to homotopy.

 \bigcirc

Role of the UIP Axiom

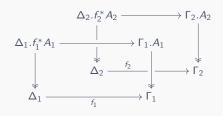




By UIP, if $w, w' \colon X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

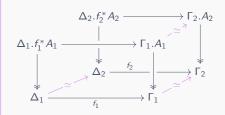


 $\overset{*}{\sim}$ Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.





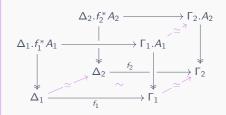
 $\overset{*}{\simeq}$ Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.



If the solid maps above are in \mathcal{W}_{ETT}



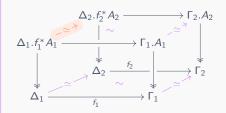
 $\overset{\text{\tiny W}}{\sim}$ Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.



If the solid maps above are in \mathcal{W}_{ETT} and bottom face commute up to homotopy



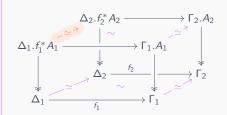
 $\overset{\text{\tiny W}}{\sim}$ Proof. \mathcal{W}_{ETT} is a class of maps defined inductively.



If the solid maps above are in W_{ETT} and bottom face commute up to homotopy then induced map is in W_{ETT} .



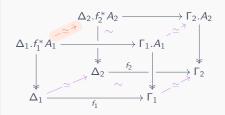
 $\overset{\text{\tiny "Proof.}}{\sim} \mathcal{W}_{\text{ETT}}$ is a class of maps defined inductively.



Inductively: parallel purple maps are homotopic.



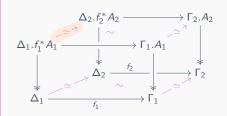
 $\overset{\text{\tiny "}}{\sim}$ Proof. \mathcal{W}_{ETT} is a class of maps defined inductively.



Inductively: parallel purple maps are homotopic. Show parallel orange maps are homotopic.



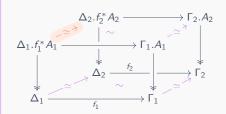
 $\overset{\sim}{\sim}$ Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.



Inductively: parallel purple maps are homotopic. Show parallel orange maps are homotopic. Construction is homotopy invariant.



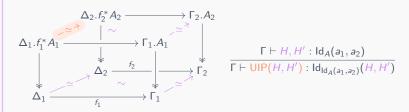
 $\overset{\sim}{\sim}$ Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.



Show that any two homotopies H,H^{\prime} for the bottom face are homotopic.



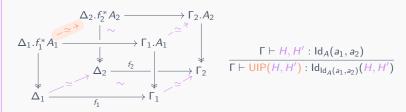
 $\overset{\sim}{\sim}$ Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.



Show that any two homotopies H, H' for the bottom face are homotopic. Homotopies are equality proofs.



 $\overset{\text{\tiny W}}{\sim}$ Proof. $\mathcal{W}_{\mathsf{ETT}}$ is a class of maps defined inductively.



Show that any two homotopies H, H' for the bottom face are homotopic. Homotopies are equality proofs. Follows by UIP.



H - Lemma

By UIP, if $w, w' \colon X \simeq X' \in \mathcal{W}_{\mathsf{ETT}}$ then $w \simeq w'$.

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H _ Lemma
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By UIP, if $w, w' : X \simeq X' \in \mathcal{W}_{ETT}$ then $w \simeq w'$.

If $\mathbb{C} \in \mathsf{CxlCat}_{\mathsf{ITT}+\mathsf{UIP}}$ cellular then the quotient category $\langle \mathbb{C} \rangle \in \mathsf{CxlCat}_{\mathsf{ETT}}$ is a category with well-defined composition.

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Theorem

The type theories ITT+UIP and ETT are Morita equivalent.

$$CxICat_{ITT+UIP} \xrightarrow{\langle - \rangle} CxICat_{ETT}$$

- **Future** directions
- Constructive proof of Hofmann's result.
- Encompassing internal universes.
- ► Further instances of Morita equivalence.

Conclusion 19/19



- Constructive proof of Hofmann's result.
- ► Encompassing internal universes.
- ► Further instances of Morita equivalence.

Thank you!