Exploring dualities beyond sound doctrines

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Gabriel-Ulmer duality

The 2-functor

$$\mathbf{Lex} \longrightarrow \mathbf{LFP}^{\mathrm{op}}$$

given by sending

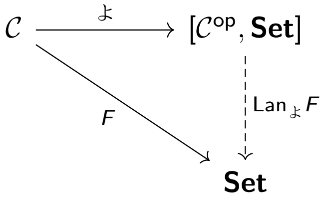
$$\mathcal{C} \longmapsto \mathbf{Lex}[\mathcal{C}, \mathbf{Set}]$$

yields an equivalence of 2-categories.

- **Lex** = 2-category of finitely complete small categories, and
- **LFP** = 2-category of locally finitely presentable categories.

Key Property (of finite limits)

Given $\mathcal{C} \in \mathbf{Lex}$, $F \colon \mathcal{C} \to \mathbf{Set}$ lex, *$Lan_{\text{よ}} F$ is also lex*:



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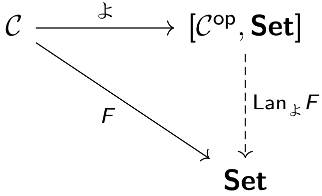

We say ("the doctrine of") finite limits is **sound**. (Important consequence: $G \in [\mathcal{C}, \mathbf{Set}]$ is lex iff it is a *filtered colimit* of representables).

Gabriel-Ulmer for sound doctrines

If Φ is a different "limit doctrine"[1] enjoying *Key Property*, then

$$\mathbf{\Phi\text{-cat}} \xrightarrow{\mathcal{C} \mapsto \Phi[\mathcal{C}, \mathbf{Set}]} \mathbf{L\Phi P}^{\mathrm{op}}$$

is an equivalence of 2-categories[2].

[1] Class of small shapes

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- $\mathbf{\Phi\text{-cat}}$: 2-category of small Φ-complete ($+$ Cauchy complete) categories, ...
- $\mathbf{L\Phi P}$: 2-category of locally Φ-presentable categories: that is, locally small, cocomplete, and generated by Φ-presentable objects under Φ-filtered colimits (a shape $\mathcal{D}$ being Φ-*filtered* if $\mathcal{D}$-colimits commute with Φ-limits in $\mathbf{Set}$).

For instance: $\Phi =$ finite limits, countable limits, finite products, no limits at all, ...

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Do we need Φ sound for such a theorem?

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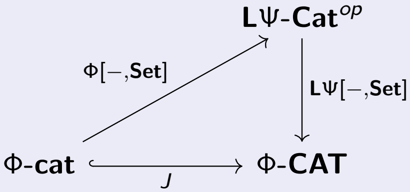
Theorem

Letting **LΨ-Cat** *be the 2-category of locally small, complete, Φ-filtered-cocomplete categories, there is a (relative) adjunction* $\Phi[-, \mathbf{Set}] \dashv_J \mathbf{L}\Psi[-, \mathbf{Set}]$.



Theorem

Letting $\mathbf{L\Psi\text{-}Cat}$ *be the 2-category of locally small, complete,* Φ*-filtered-cocomplete categories, there is a (relative) adjunction* $\Phi[-,\mathbf{Set}] \dashv_J \mathbf{L\Psi}[-,\mathbf{Set}]$.

$$\begin{array}{ccc} & & \mathbf{L\Psi\text{-}Cat}^{op} \\ & \overset{\Phi[-,\mathbf{Set}]}{\nearrow} & \Big\downarrow{\scriptstyle \mathbf{L\Psi}[-,\mathbf{Set}]} \\ \Phi\text{-}\mathbf{cat} & \underset{J}{\hookrightarrow} & \Phi\text{-}\mathbf{CAT} \end{array}$$

Theorem

If $\Phi[-,\mathbf{Set}] \colon \Phi\text{-}\mathbf{cat} \to \mathbf{L\Phi P}^{op}$ *is an equivalence, then* Φ *is sound.*

Some limit doctrines are not sound (e.g. pullbacks, countable products, ...), but we still want to be able to recover

$$\mathcal{C} \in \mathbf{\Phi\text{-cat}}$$

from

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Towards an understanding of the general situation, we investigate whether $\Phi[-, \mathbf{Set}]$ *reflects equivalences*.

Theorem

(Under technical assumption on Φ*)* $\Phi[-, \mathbf{Set}]$ *reflects equivalences* $\iff$ $\mathbf{\Phi\text{-cat}}$ *has the property that for 1-cells, fully faithful* $+$ *lax epimorphism* $\implies$ *equivalence.*

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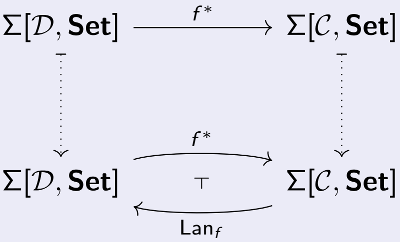
Brief comment on proof.

($\implies$) Assumption on Φ gives us a larger, sound doctrine $\Sigma \supset \Phi$ with $\mathbf{L\Phi P} \simeq \mathbf{L\Sigma P}$. Can show property holds of Σ exploiting fact that Σ**-cat** $\to$ **cat** sends arrows to right adjoint arrows, via *Key Property*.



(Then show that property passes to $\Phi \subset \Sigma$). □

Moreover[3],

Lemma

*A 1-cell f in Φ-**cat** is a lax epimorphism $\iff$ it is Φ-absolutely codense.*

Together:

Theorem

$\Phi[-, \mathbf{Set}]$ *reflects equivalences* $\iff$ *a functor is Φ-absolutely codense only if it is essentially surjective.*

("All Φ-complete categories are Φ-absolutely complete").

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("All Φ-complete categories are Φ-absolutely complete"). In summary:

- To get a Gabriel-Ulmer-style duality $\Phi[-,\mathbf{Set}]\colon \Phi\text{-}\mathbf{cat} \to \mathbf{L\Phi P}^{\mathrm{op}}$, we need that Φ is sound.
- ...but, so long as there are no interesting Φ-absolute limits, we can at least say that $\Phi[-,\mathbf{Set}]$ reflects equivalences.

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