Categories or Spaces?

Categorical Concepts in Noncommutative Geometry

Wendy Lowen

j.w. Violeta Borges Marques, Lander Hermans, Arne Mertens

Universiteit Antwerpen

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categories

Mirror symmetry

Mirror symmetry has originally been observed for Calabi-Yau (CY) manifolds. For two n-dimensional mirror manifolds X and Y, we in particular have:

$$h^{p,q}(X) = h^{n-p,q}(Y)$$

where $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$ are the *hodge numbers* of a complex manifold X.

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For a CY 3-fold X:

(A) $h^{1,1}(X)$ is related to symplectic deformations

(B) $h^{2,1}(X)$ is related to complex deformations

Hence, for mirror CY 3-folds X and Y, complex deformations of X correspond to symplectic deformations of Y.

Homological Mirror Symmetry (HMS)

In his 1994 ICM address, Kontsevich made the following conjectural proposal:

Define X and Y to satisfy HMS provided we have (exact) equivalences of (triangulated) categories:

$$D(\mathrm{Qch}(X)) \cong D(\mathcal{F}(Y)) \text{ and } D(\mathrm{Qch}(Y)) \cong D(\mathcal{F}(X))$$

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2. HMS takes place in an extended realm of certain "noncommutative spaces" stemming from more general deformations

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2. HMS takes place in an extended realm of certain "noncommutative spaces" stemming from more general deformations

$\rightsquigarrow$ *look at categorical invariants!*

Hochschild cohomology

X scheme (quasi-compact, separated)

How should we deform X?

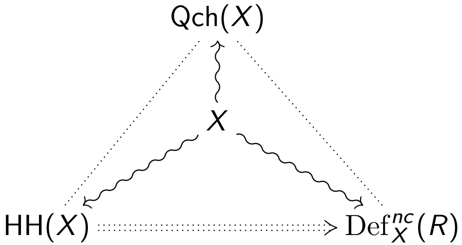
- $\mathrm{HH}^n(X) = \mathrm{Ext}^n_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$ (Swan, 1996)
- HKR (smooth case): $\mathrm{HH}^n(X) = \oplus_{p+q=n}\mathrm{H}^p(X, \Lambda^q\mathcal{T}_X)$

$$\mathrm{HH}^2(X) = \mathrm{H}^0(X, \Lambda^2\mathcal{T}_X) \oplus \mathrm{H}^1(X, \mathcal{T}_X) \oplus \mathrm{H}^2(X, \mathcal{O}_X)$$

- $\mathrm{H}^1(X, \mathcal{T}_X) \leftrightarrow$ first order scheme deformations

Noncommutative spaces?

X a "noncommutative space"



$\rightsquigarrow$ *associate algebraic objects to a scheme and then deform*

Affine schemes

$X = \mathrm{Spec}(A)$

A commutative k-algebra

- $\mathsf{Qch}(X) = \mathsf{Mod}(A)$
- Attempt: realise $\mathrm{HH}^2(X)$ by deforming A

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Key example: $X = \mathbb{A}^2 = \mathrm{Spec}(k[x, y])$

$\rightsquigarrow$ deforms into the Weyl algebra:

$$k\langle x, y\rangle / xy - yx - \lambda$$

- $\mathsf{HH}^n(\mathrm{Spec}(A)) \cong \mathsf{HH}^n(A) = \mathsf{Ext}^n_{A-A}(A, A)$, the Hochschild cohomology of A (Hochschild, 1945)

Deligne's principle

"Every deformation problem is governed by a dg Lie algebra (DGLA)" (Deligne, 1986)

Let $(L, [-,-], d)$ be a DGLA. Consider the *Maurer-Cartan equation*

$$\mathrm{MC}(\phi) = d(\phi) + \frac{1}{2}[\phi, \phi].$$

There is an associated *deformation functor* $\mathrm{Def}_L : \mathrm{Art}_k \longrightarrow \mathrm{Set}$ with

$$\mathrm{Def}_L(R, \mathfrak{m}) = \{\phi \in (\mathfrak{m} \otimes L)^1 \mid \mathrm{MC}(\phi) = 0\}/\sim$$

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Remark: DGLA's correspond precisely to "formal moduli problems" in the setup of derived algebraic geometry (Lurie and Pridham, 2010).

Algebraic deformation theory

Let A be a k-vector space and put $\mathbf{C}^n(A) = \mathrm{Hom}_k(A^{\otimes n}, A)$.

$\rightsquigarrow$ *operadic composition* entailing the braces, e.g.

$$\phi \bullet \psi = \sum (-1)^{\epsilon} \phi \circ (1 \otimes \ldots \psi \cdots \otimes 1)$$

Put

$$[\phi, \psi] = \phi \bullet \psi - (-1)^{|\phi||\psi|} \psi \bullet \phi.$$

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Put

$$[\phi, \psi] = \phi \bullet \psi - (-1)^{|\phi||\psi|} \psi \bullet \phi.$$

Then $(\mathbf{C}(A)[1], [-,-], 0)$ is a DGLA such that for $m \in \mathrm{Hom}_k(A^{\otimes 2}, A)$ we have

$$\mathrm{MC}(m) = m \bullet m = m \circ (m \otimes 1) - m \circ (1 \otimes m)$$

whence

$$\mathrm{MC}(m) = 0 \iff m \text{ is associative.}$$

Algebraic deformation theory

Let (A, m) be a k-algebra and consider $\mathbf{C}(A)$.
We obtain a differential $d_{Hoch} = [m, -]$, with eg.

$$d_{Hoch}(\phi)(a, b, c) = a\phi(b, c) - \phi(ab, c) + \phi(a, bc) - \phi(a, b)c$$

for $\phi \in \mathbf{C}^2(A) = \mathrm{Hom}_k(A^{\otimes 2}, A)$, such that $\mathrm{HH}^n(A) = H^n\mathbf{C}(A)$.

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Definition (Gerstenhaber, 1964)

Let A be a k-algebra and let R be an Artin local k-algebra. An *R-deformation* of A is a flat R-algebra $\bar{A}$ with an isomorphism $k \otimes_R \bar{A} \cong A$.

Then $L = (\mathbf{C}(A)[1], [-, -], d_{Hoch})$ is a DGLA with

$$\mathrm{Def}_L \cong \mathrm{Def}_A^{alg}.$$

Algebraic deformation theory

Algebraic deformation theory

Example

Put $R = k[\epsilon] = k[t]/(t^2)$. Then $\mathrm{Def}_L(k[\epsilon]) \cong \mathsf{HH}^2(A)$ and

$$\phi \in \mathsf{Z}^2\mathbf{C}(A) \longmapsto (A \oplus A\epsilon, \bar{m} = m + \phi\epsilon)$$

yields $\mathsf{HH}^2(A) \cong \mathrm{Def}_A(k[\epsilon])$. For $A = k[x, y]$, we obtain $k[\epsilon][x, y]$ with

$$\bar{m}(f, g) = fg + h\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\epsilon$$

for some $h \in k[x, y]$.

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Observation: if $k \otimes_R \bar{A} \cong A$, we have

$$\mathsf{Fun}_R(k, \mathsf{Mod}(\bar{A})) \cong \mathsf{Mod}(A).$$

$\rightsquigarrow$ Deformation theory of abelian categories (L - Van den Bergh, 2005).

Part 1: linear...

topoi

Projective schemes

$X = \mathrm{Proj}(A)$

$A = (A_i)_{i \in \mathbb{Z}}$ positively graded, connected commutative k-algebra

- Serre's Theorem: $\mathsf{Qch}(X) = \mathsf{Qgr}(A)$
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Key example: $X = \mathbb{P}^2 = \mathrm{Proj}(k[x_0, x_1, x_2])$
Noncommutative $\mathbb{P}^2$'s = Sklyanin algebras

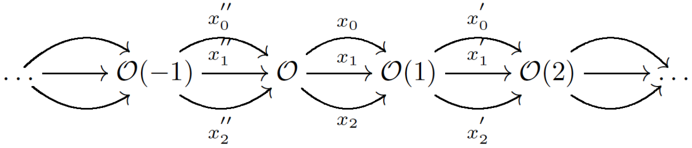
$$k\langle x_0, x_1, x_2\rangle / (cx_i^2 + bx_{i+1}x_{i+2} + ax_{i+2}x_{i+1})_{i \in \mathbb{Z}_3}$$

(Artin - Tate - Van den Bergh, 1990, Bondal - Polishchuk, 1993)

Projective schemes

X projective scheme (eg. $\mathbb{P}^2$)

$\rightsquigarrow$ $\mathbb{Z}$-algebra $\mathfrak{a}$ (linear category with objects indexed by $\mathbb{Z}$)



There is a linear *tails* topology on $\mathfrak{a}$ with

$$\mathsf{Qch}(X) \cong \mathsf{Mod}(\mathfrak{a})/\mathsf{Tors}(\mathfrak{a}) \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\text{tails}})$$

$\rightsquigarrow$ $\mathcal{T}_{\text{tails}}$ is a linearisation of the Grothendieck topology on $(\mathbb{Z}, \geq)$ for which all non-empty sieves are covering

$\mathbb{Z}$-algebras

X projective with ample invertible line bundle $\mathcal{L}$ and

$$H^1(X, \mathcal{O}_X) = 0 = H^2(X, \mathcal{O}_X) \tag{1}$$

There is a $\mathbb{Z}$-algebra $\mathfrak{a}$ on the $\mathcal{L}^n$ with $\mathrm{HH}^n(X) = \mathrm{HH}^n(\mathfrak{a})$ (Van den Bergh, 2001; L - Van den Bergh, 2005; L, 2012)

$\rightsquigarrow$ *deform* $\mathfrak{a}$ *algebraically and use* $\mathcal{T}_{\mathrm{tails}}$ *to construct geometry!*

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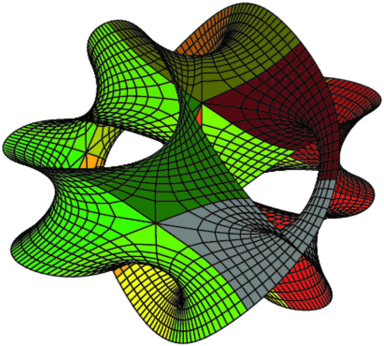
$\rightsquigarrow$ *deform* $\mathfrak{a}$ *algebraically and use* $\mathcal{T}_{\mathrm{tails}}$ *to construct geometry!*

$\rightsquigarrow$ HMS has been extended to Del Pezzo surfaces and their noncommutative deformations (Auroux - Katzarkov - Orlov, 2005)

Question: what about schemes that do not satisfy (1)?

The quartic



K3 surface X cut out by $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ in $\mathbb{P}^3$, which has $\dim(H^2(X, \mathcal{O}_X)) = h^{0,2} = 1$.

Linear topologies

A Grothendieck category is a cocomplete abelian category with a generator and exact filtered colimits.

- Every Grothendieck category can be represented as a linear sheaf category (Gabriel - Popescu)
- Grothendieck categories are stable under the tensor product of linear locally presentable categories (L - Ramos González - Shoikhet, 2017)

$$\mathsf{Sh}(\mathfrak{a}_1, \mathcal{T}_1) \boxtimes \mathsf{Sh}(\mathfrak{a}_2, \mathcal{T}_2) = \mathsf{Sh}(\mathfrak{a}_1 \otimes \mathfrak{a}_2, \mathcal{T}_1 \boxtimes \mathcal{T}_2)$$

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- The Grothendieck property is stable under abelian deformation (L - Van den Bergh, 2005), but a given site may not deform algebraically!

Schemes

X scheme (quasi-compact, separated)

- $\mathrm{HH}^n(X) = \mathrm{Ext}^n_{X\times X}(\mathcal{O}_X, \mathcal{O}_X)$
- HKR (smooth case): $\mathrm{HH}^n(X) = \oplus_{p+q=n}\mathrm{H}^p(X, \Lambda^q\mathcal{T}_X)$

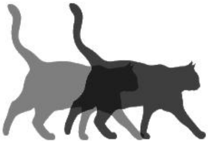
$$\mathrm{HH}^2(X) = \mathrm{H}^0(X, \Lambda^2\mathcal{T}_X) \oplus \mathrm{H}^1(X, \mathcal{T}_X) \oplus \mathrm{H}^2(X, \mathcal{O}_X)$$

$\rightsquigarrow$ a class $u = (\gamma, \beta, \alpha)$ on the right determines an abelian deformation $\mathrm{Qch}(X, u)$ of $\mathrm{Qch}(X)$ (Toda, 2009; Dinh Van - Liu - L, 2017)

$\rightsquigarrow$ derived categories of twisted sheaves (Căldăraru, 2000)

$\rightsquigarrow$ *higher order deformations?*

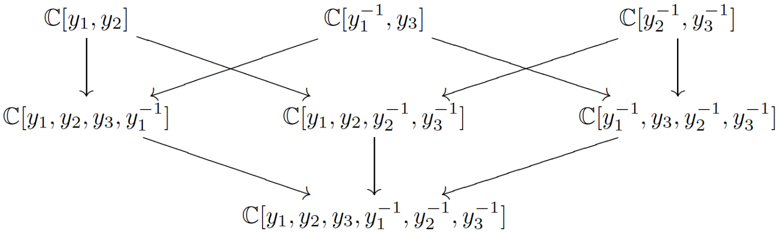
virtual double categories



Schemes

X quasi-compact separated scheme (eg. $X = \mathbb{P}^2$)

$\rightsquigarrow$ structure sheaf $\mathbb{A} = \mathcal{O}_X|_{\mathcal{U}}$ on affine cover $\mathcal{U}$



- Qch(X) can be reconstructed from $\mathbb{A}$
- $\mathrm{HH}^*(X) \cong \mathrm{H}^*\mathbf{C}_{GS}(\mathbb{A})$

  (Gerstenhaber - Schack, 1983; L - Van den Bergh, 2005)

Presheaves of algebras

(A, m, f) presheaf of k-algebras on small category $\mathcal{U}$ $(A : U \mapsto A_U)$

$\rightsquigarrow$ associated Gerstenhaber-Schack complex $\mathbf{C}_{GS}(A)$

$$\mathbf{C}^{p,q}_{GS}(A) = \prod_{\sigma \in N_p(\mathcal{U})} \mathrm{Hom}_k(A_{t\sigma}^{\otimes q}, A_{s\sigma})$$

The total differential d_{GS} is built from

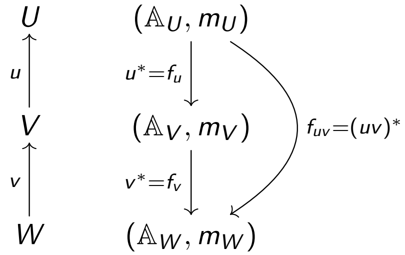
- horizontal Hochschild differentials d_{Hoch}
- vertical simplicial differentials d_{simp}

Components of total degree two:

- $\mathbf{C}^{0,2}_{GS}(A) = \prod_{U \in \mathcal{U}} \mathrm{Hom}_k(A_U \otimes A_U, A_U) \ni m$
- $\mathbf{C}^{1,1}_{GS}(A) = \prod_{u : V \to U} \mathrm{Hom}_k(A_U, A_V) \ni f$
- $\mathbf{C}^{2,0}_{GS}(A) = \prod_{(v : W \to V, u : V \to U)} A_W$

Prestacks

$(\mathbb{A}, m, f, c)$ prestack on $\mathcal{U}$



$$m_U : \mathbb{A}_U \otimes \mathbb{A}_U \to \mathbb{A}_U$$

$$f_u : \mathbb{A}_U \to \mathbb{A}_V$$

$$c^A_{u,v} \in \mathrm{Hom}_k(v^* u^*(A), (uv)^*(A))$$

Prestacks: axioms

$(\mathbb{A}, m, f, c)$ prestack on $\mathcal{U}$

$$\mathbf{C}^{p,q}_{GS}(\mathbb{A}) = \prod_{\sigma \in N_p(\mathcal{U}),\, A \in \mathbb{A}^{q+1}_{t\sigma}} \mathrm{Hom}_k(\mathbb{A}^{\otimes q}_{t\sigma}(A), \mathbb{A}_{s\sigma}(\sigma^* sA, |\sigma|^* tA))$$

- $\mathbf{C}^{0,3}_{GS}(A)$: associativity of m: $m \circ (m \otimes 1) = m \circ (1 \otimes m)$
- $\mathbf{C}^{1,2}_{GS}(A)$: functoriality of f: $f \circ m = m \circ (f \otimes f)$
- $\mathbf{C}^{2,1}_{GS}(A)$: naturality of c: $m \circ ((f \circ f) \otimes c) = m \circ (c \otimes f)$
- $\mathbf{C}^{3,0}_{GS}(A)$: coherence: $m \circ (c \otimes c) = m \circ ((f \circ c) \otimes c)$

Prestacks: axioms

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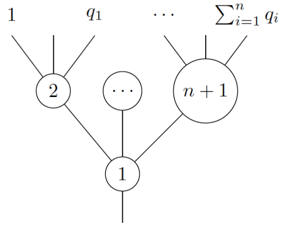
$\rightsquigarrow$ *relations are not quadratic!*

Algebras: operadic structure

Recall that there is an $\mathbb{N}$-coloured operad Op whose algebras are precisely nonsymmetric operads.

- Op is generated by



- elements of $\mathrm{Op}(k)$ can be depicted as trees with k vertices.

- Op acts on $\mathbf{C}(A)$ of an algebra A by inserting operations of designated arities at vertices, and composing.

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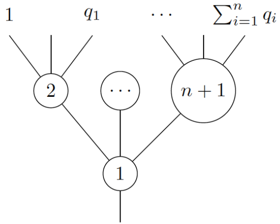

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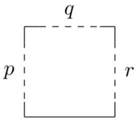
- Op acts on $\mathbf{C}(A)$ of an algebra A by inserting operations of designated arities at vertices, and composing.

$\rightsquigarrow$ *let a similar coloured operad act on* $\mathbf{C}_{GS}(\mathbb{A})$

Prestacks: box operadic structure

We define an $\mathbb{N}^3$-coloured operad $\Box$p (pronounced "box-op")

- the colour (p, q, r) $\leftrightarrow$ the box 

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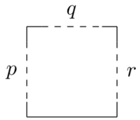

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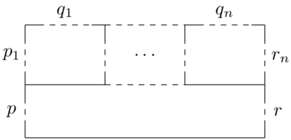
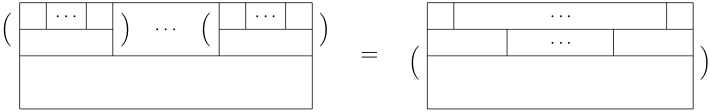
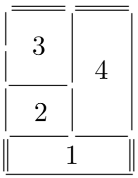


with associativity relations



Prestacks: box operadic structure

- elements of $\square\mathsf{p}(n)$ can be depicted as *n-stackings*, that is trees with n matching (p, q, r)-labeled boxes as vertices. E.g:



$\in \square\mathsf{p}(4)$

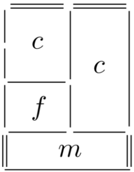
assembles boxes with labels $(0, 2, 0)$, $(1, 1, 1)$, $(2, 0, 1)$ and $(2, 0, 1)$ respectively into a $(3, 0, 1)$-box.

Prestacks: box operadic structure

The operad □p acts on an enlargement $\mathbf{C}_{\square}(\mathbb{A})$ of $\mathbf{C}_{GS}(\mathbb{A})$ with

$$\mathbf{C}_{\square}^{p,q,r}(\mathbb{A}) = \prod_{\substack{\sigma \in N_p(\mathcal{U}),\, h \in \boldsymbol{\Delta}_f([r],[p]) \\ A \in \mathbb{A}(t\sigma)^{q+1}}} \mathrm{Hom}_k(\mathbb{A}(t\sigma)^{\otimes q}(A), \mathbb{A}(s\sigma)(\sigma^* sA, h(\sigma)^* tA))$$

by inserting linear maps into rectangles, and composing:



$\rightsquigarrow \quad m \circ ((f \circ c) \otimes c)$

L_∞-structure

We totalise $k\Box\mathrm{p}$ into a graded operad $\Box\mathrm{p}_{\mathrm{gr}}$. Let $\Box\mathrm{p}_{\mathrm{grt}}^{2-n}(n)$ be the set of n-stackings of degree $2-n$ + technical assumptions.

For $n \geq 2$, we define the element $P_n \in \Box\mathrm{p}_{\mathrm{gr}}(n)$ as

$$P_n = \sum_{S \in \Box\mathrm{p}_{\mathrm{grt}}^{2-n}(n)} (-1)^S S$$

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and the *n-Gerstenhaber bracket* L_n as the anti-symmetrisation

$$L_n = \sum_{\sigma \in \mathbb{S}_n} (-1)^\sigma L_n^\sigma$$

Theorem (Dinh Van - Hermans - L)

We have a morphism of dg-operads $L_\infty \to \square\mathrm{p}_{\mathrm{gr}} : l_n \mapsto L_n$.

Box operads

In analogy with nonsymmetric operads being Op-algebras, we introduce the following terminology:

Definition

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A $\square$p-algebra is called a *box operad*.

Rephrasing the theorem, we have shown that every linear box operad $\mathcal{B}$ carries an L_∞-structure (with zero differential). The Maurer-Cartan equation takes the following form, for $\alpha \in \mathcal{B}$:

$$\mathrm{MC}(\alpha) = \sum_{n \geq 2} (-1)^{\frac{n(n+1)}{2}} P_n(\alpha, \ldots, \alpha)$$

Proposition

The resulting L_∞-structure on $\mathbf{C}_\square(\mathbb{A})$ restricts to an L_∞-structure on $\mathbf{C}_{GS}(\mathbb{A})$.

Historical notes

- Box operads are an instance of multicategories over a monad (Burroni, 1971) and have been called fc multicategories (Leinster, 1999, 2003). More recently they are being studied under the name of virtual double categories (Crutwell - Shulman, 2010; Koudenburg, 2020, ...).

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- In specific cases, L_∞-structures on $\mathbf{C}_{GS}(\mathbb{A})$ were obtained by other methods, for instance for an algebra morphism (Frégier - Markl - Yau, 2009) and for specific diagrams of algebras (Barmeier - Frégier, 2018). The case of a general presheaf of algebras was solved by Hawkins (2020) and extended to prestacks by Dinh Van - L - Hermans (2022).
  However, these approaches do not allow for a characterisation of the prestack structure.

Prestacks: box operadic structure

Let $\mathbb{A}$ be a k-quiver on $\mathcal{U}$ (i.e. a prestack without the algebraic structure).

Theorem (Dinh Van - Hermans - L)

Let $\mathbf{C}_{GS}(\mathbb{A})$ *be endowed with the box operadic* L_∞*-structure. Consider* $\alpha = (m, f, c) \in \mathbf{C}^2_{GS}(\mathbb{A})$. *We have*

$$\mathrm{MC}(\alpha) = 0 \iff (\mathbb{A}, m, f, c) \text{ is a prestack.}$$

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Corollary

Let $(\mathbb{A}, m, f, c)$ be a prestack. The deformation theory of $\mathbb{A}$ as a prestack is governed by the box operadic L_∞-structure on $\mathbf{C}_{GS}(\mathbb{A})$ twisted by $\alpha = (m, f, c)$.

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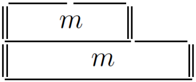
$\rightsquigarrow$ A minimal model for prestack via Koszul duality for box operads (Hermans, 2023)

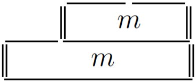
Prestacks: box operadic structure

Proof.

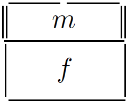
$\mathrm{MC}(\alpha) = -P_2(\alpha,\alpha) + P_3(\alpha,\alpha,\alpha) + P_4(\alpha,\alpha,\alpha,\alpha).$

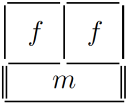
$\mathrm{MC}(\alpha)_{[0,3]} = -P_2(\alpha,\alpha)_{[0,3]}$

$= -$ 

$+$ 

$\mathrm{MC}(\alpha)_{[1,2]} = -P_2^{GS}(\alpha,\alpha)_{[1,2]} + P_3^{GS}(\alpha,\alpha,\alpha)_{[1,2]}$

$=$ 

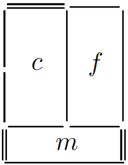
$-$ 

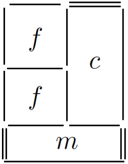
□

Prestacks: box operadic structure

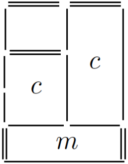
Proof.

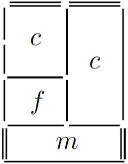
$$\mathrm{MC}(\alpha)_{[2,1]} = P_3^{GS}(\alpha,\alpha,\alpha)_{[2,1]} + P_4^{GS}(\alpha,\alpha,\alpha,\alpha)_{[2,1]}$$

$=$ 

$-$ 

$$\mathrm{MC}(\alpha)_{[3,0]} = P_3^{GS}(\alpha,\alpha,\alpha)_{[3,0]} + P_4^{GS}(\alpha,\alpha,\alpha,\alpha)_{[3,0]}$$

$= -$ 

$+$ 

Mirror symmetry

Mirror picture:

$$B: \quad X \dashrightarrow D(\mathrm{Qch}(X)) \cong D(\mathcal{F}(Y)) \dashleftarrow Y \quad : A$$

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Compelling reasons to deform dg categories:

1. $\mathcal{F}(X)$ is an A_∞-category
2. $D(\mathrm{Qch}(X)) \cong D(A)$ for a dg algebra A (Keller, 1994; Neeman, 1996; Bondal - Van den Bergh, 2003)
3. Mirror symmetry involves dg categories on the B-side without abelian models (Orlov, 2003)

Dg categories

Problem: deformation theory of dg categories is notoriously difficult due to "curvature" (Keller - Lowen, 2009; Lurie, 2010; Lehmann, 2024).

Inspiration:

1. dg categories as higher categories:

$$\{\text{pretriangulated dg cats}\} \leftrightarrow \{\text{stable linear } \infty\text{-cats}\}$$

   (Lurie, 2016, Cohn, 2016)
2. general theory of enriched ∞-categories (Gepner - Haugseng, 2015)

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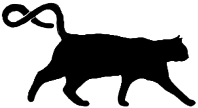
$\rightsquigarrow$ *establish a concrete model of linear ∞-categories amenable to algebraic deformation theory*

Part 3: linear...

∞-categories



Quasi-categories in modules

"Quasi-categories in $\mathcal{V}$ are ∞-categories weakly enriched in $S\mathcal{V}$"

$\mathcal{V} = \mathrm{Set} \rightsquigarrow \mathcal{V} = \mathrm{Mod}(k)$; $\mathrm{SSet} \rightsquigarrow S\,\mathrm{Mod}(k) \cong C(k)_{\geq 0}$

Goals: develop their

- homotopy theory (Arne Mertens)
- deformation theory $\rightsquigarrow$ *today*

First step: introduce an appropriate ambient category $S_{\otimes}\mathcal{V}$ of *templicial objects* or *tensor-simplicial objects*

Enriched nerve

Let $\mathcal{C}$ be a small k-linear category. Consider the k-modules

$$N_k(\mathcal{C})_n = \bigoplus_{A_0,\dots,A_n \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(A_0, A_1) \otimes \dots \otimes \mathcal{C}(A_{n-1}, A_n)$$

$$u = f_1 \otimes \cdots \otimes f_n \in \mathcal{C}(A_0, A_1) \otimes \dots \otimes \mathcal{C}(A_{n-1}, A_n)$$

- $d_i(u) = f_1 \otimes \cdots \otimes f_{i+1} f_i \otimes \cdots \otimes f_n$ for $1 \leq i \leq n-1$

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- $d_i(u) = f_1 \otimes \cdots \otimes f_{i+1} f_i \otimes \cdots \otimes f_n$ for $1 \leq i \leq n-1$
- $d_0(u) = ?$ $\quad d_n(u) = ?$

Problem: the $N_k(\mathcal{C})_n$ do not constitute a simplicial k-module

Enriched nerve

Solution: restrict $\mathbf{\Delta}$ to the *finite interval category* $\mathbf{\Delta}_f$:

- objects: the posets $[n] = \{0, \ldots, n\}$ with $n \geq 0$
- order morphisms $f : [n] \to [m]$ with $f(0) = 0$ and $f(n) = m$

The category $\mathbf{\Delta}_f$ is strict monoidal with $[n] + [m] = [n + m]$ and $[0]$ as tensor unit.

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Proposition (Leinster, 2000)

Let $(\mathcal{V}, \times, 1)$ *be a cartesian monoidal category. There is an isomorphism of categories*

$$\mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathcal{V}) \cong S\mathcal{V}.$$

In particular, we have $\mathrm{Colax}(\mathbf{\Delta}_f^{op}, \mathrm{Set}) \cong \mathrm{SSet}$.

Templicial objects

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category and O a set. A *$\mathcal{V}$-quiver on vertex set O* consists of $\mathcal{V}$-objects $Q(a, b)$ for $a, b \in O$. The category $\mathcal{V}\,\mathrm{Quiv}_O$ of $\mathcal{V}$-quivers on O is monoidal with

$$(Q \otimes_O P)(a, b) = \coprod_{c \in O} Q(a, c) \otimes P(c, b) \quad \text{and} \quad I_O(a, b) = \begin{cases} I & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

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Definition

A *templicial object in $(\mathcal{V}, \otimes, I)$ with vertex set O* is a strongly unital, colax monoidal functor

$$X : \mathbf{\Delta}_f^{op} \to \mathcal{V}\,\mathrm{Quiv}_O\,.$$

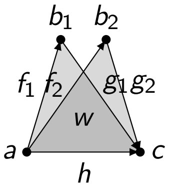
The category of templicial objects in $\mathcal{V}$ is denoted by $S_{\otimes}\mathcal{V}$.

$\rightsquigarrow$ Discrete vertices in simplicial objects internal to a monoidal category (Mertens, 2025)

Templicial objects

Example

Let $\mathcal{V} = \text{Mod}(k)$. Consider the templicial vector space X:



$$
\begin{aligned}
f_1 &\in X_1(a, b_1), \quad g_1 \in X_1(b_1, c) \\
f_2 &\in X_1(a, b_2), \quad g_2 \in X_1(b_2, c) \\
h &\in X_1(a, c), \quad w \in X_2(a, c)
\end{aligned}
$$

with $d_1(w) = h$ and $\mu_{1,1}(w) = f_1 \otimes g_1 + f_2 \otimes g_2$.

Necklaces

Let $X : \mathbf{\Delta}_f^{\mathrm{op}} \longrightarrow \mathcal{V}\,\mathrm{Quiv}_O$ be a templicial object in $\mathcal{V}$.
For $a, b \in O$, the functor $X_\bullet(a, b) : \mathbf{\Delta}_f^{\mathrm{op}} \longrightarrow \mathcal{V}$ can naturally be extended to a functor

$$X_\bullet(a, b) : \mathcal{N}ec^{\mathrm{op}} \longrightarrow \mathcal{V}$$

determined by

$$X_{\Delta^{n_1} \vee \dots \vee \Delta^{n_k}}(a, b) = X_{n_1} \otimes_O \cdots \otimes_O X_{n_k}(a, b)$$

on objects and

$$\begin{array}{rcl} \delta^j : \Delta^{n-1} \to \Delta^n & \mapsto & d_j : X_n \to X_{n-1} \\ \sigma^i : \Delta^n \to \Delta^{n-1} & \mapsto & s_i : X_{n-1} \to X_n \\ \nu^{p,q} : \Delta^p \vee \Delta^q \to \Delta^{p+q} & \mapsto & \mu_{p,q} : X_{p+q} \to X_p \otimes X_q \end{array}$$

on morphisms.

Quasi-categories in $\mathcal{V}$

Definition

Let $Y : \mathcal{N}ec^{op} \longrightarrow \mathcal{V}$ be a functor. We say that Y is *weak Kan* if for all $0 < j < n$ any lifting problem

$$\begin{array}{ccc} \tilde{F}(\Lambda^n_j)_\bullet(0,n) & \longrightarrow & Y \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ \tilde{F}(\Delta^n)_\bullet(0,n) & & \end{array}$$

in $\mathrm{Fun}(\mathcal{N}ec^{op}, \mathcal{V})$ has a solution.

We call a templicial object X a *quasi-category in $\mathcal{V}$* if the functors $X_\bullet(a,b)$ are weak Kan for all $a, b \in O$.

The templicial dg nerve

Theorem (L - Mertens)

There is a templicial dg nerve N_k^{dg} from the category $\mathrm{Cat}_{dg}(k)$ of small dg-categories to the category $S_\otimes \mathrm{Mod}(k)$ of templicial modules, which lands in quasi-categories in modules:

$$\begin{array}{ccc} \mathrm{Cat}_{dg}(k) & \xrightarrow{N_k^{dg}} & S_\otimes \mathrm{Mod}(k) \\ & \searrow^{N^{dg}} & \downarrow \tilde{U} \\ & & \mathrm{SSet} \end{array}$$

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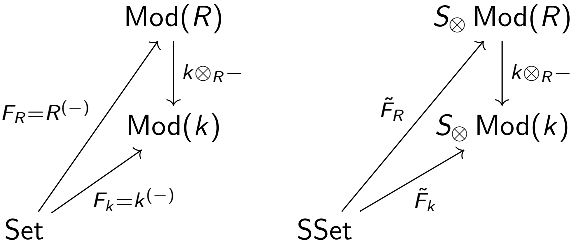
$\rightsquigarrow$ Nerves of enriched categories via necklaces (Mertens, 2024)

$\rightsquigarrow$ Templicial nerve of an A_∞-category (Borges Marques - Mertens, 2024)

Base change

Consider the following functors relating different enriching categories $\mathcal{V}$ (for R a commutative k-algebra):



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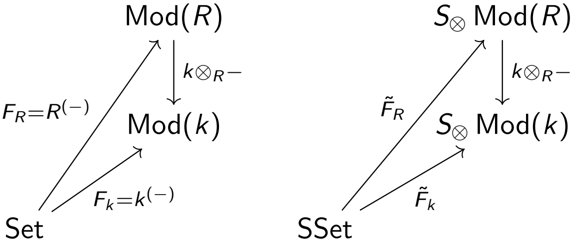


Proposition (L - Mertens)

The free functor $\tilde{F}_R$ preserves (enriched) quasi-categories.

The proof makes use of non-associative Frobenius structures and wings $W^n = \partial_0\Delta^n \cup \partial_n\Delta^n \subseteq \Delta^n$.

Deformations of templicial modules

Definition

Let R be an Artin local k-algebra. An R-deformation of a templicial k-module X is a levelwise flat templicial R-module $\bar{X}$ with $k \otimes_R \bar{X} \cong X$.

Example

Let $\bar{\mathcal{C}}$ be a (flat) R-deformation of a k-linear category $\mathcal{C}$. Then $N_R(\bar{\mathcal{C}})$ is a R-deformation of $N_k(\mathcal{C})$.

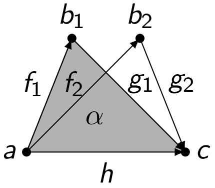
Example

Let X be a simplicial set. Then $\tilde{F}_R(X)$ is an R-deformation of $\tilde{F}_k(X)$.

Deformations of templicial modules

Example

Put $R = k[\epsilon]$ with $\epsilon^2 = 0$. We define $P = \tilde{F}(\Delta^2 \coprod_{\Delta^1} \partial\Delta^2)$ using the inclusions $\delta_1 : \Delta^1 \to \Delta^2$ and $\delta_1 : \Delta^1 \to \partial\Delta^2$ in SSet:



$$
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h \in P_1(a, c), &\quad \alpha \in P_2(a, c)
\end{aligned}
$$

with $d_1(\alpha) = h$ and $\mu_{1,1}(\alpha) = f_1 \otimes g_1$.

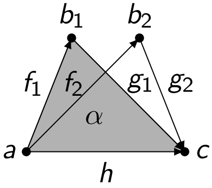
Deformations of templicial modules

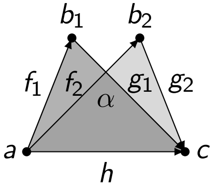
Example (continued)

We obtain a $k[\epsilon]$-deformation $\bar{P}$ of P with

$$\bar{\mu}_{1,1}(\alpha) = f_1 \otimes g_1 + f_2 \otimes g_2\epsilon$$
$$\bar{d}_1(\alpha) = h$$

A picture of P and $\bar{P}$, on the left and right, respectively:





Note that $\bar{P}$ is a non-free deformation of the free templicial module P.

Deformations of templicial modules

Theorem (Borges Marques - L - Mertens)

The quasi-category property is stable under infinitesimal deformation of templicial modules.

Deformations of templicial modules

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Theorem (Borges Marques)

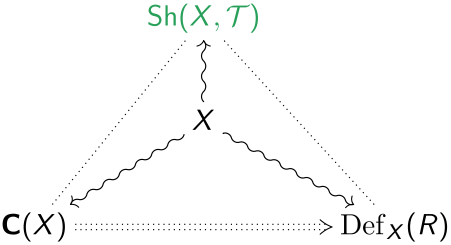
Let X be a templicial k-module. There is a Hochschild complex $\mathbf{C}(X)$ that governs infinitesimal deformations of X via an obstruction theory involving $\mathrm{HH}^{2,3}(X) = H^{2,3}\mathbf{C}(X)$.

Future goal: for $\mathcal{C}$ a cohomologically bounded above or pretriangulated dg category, establish

$$\mathbf{C}(\mathcal{C}) \cong \mathbf{C}(N_k^{dg}(\mathcal{C}))$$

Quasi-categories in modules as noncommutative spaces?

X quasi-category in modules



Future goals:

1. Develop *linear* higher topos theory to define sheaf categories
2. Use 1. in deformation theory cfr Part 1.
3. Endow $\mathbf{C}(X)$ with higher structure cfr Part 2.