Giry monad revisited

Daniel Luckhardt University College London July 18, 2024

Overview

- Up-date on my poster presentation from last year.
- Issues with extension operation.
- Giry monad only as endofunctor but with nice properties.
- Alternative set theories.

But let's start again at the beginning of the story.

Limitations of general Giry monad

General measures (denoted by $\mathfrak{m}, \mathfrak{n}, \mathfrak{m}_1, \ldots$) lack many desireable properties.

Analytic properties

No Kantorovic-Rubinstein duality

$$W(c) = K(c)$$

for every bounded measurable cost function c, where

$$W(c)(\mathfrak{m}_1, \mathfrak{m}_2) = \sup_{\mathfrak{c} \text{ couples } \mathfrak{m}_1, \mathfrak{m}_2} \int c \, \mathrm{d}\mathfrak{c}$$

($\mathfrak{c}$ couples $\mathfrak{m}_1, \mathfrak{m}_2$ if $\mathrm{pr}_{i*}\, \mathfrak{c} = \mathfrak{m}_i$ for $i = 1, 2$) and

$$K(c)(\mathfrak{m}_1, \mathfrak{m}_2) = \sup_{\substack{h \text{ nonexpansive} \\ \text{wrt. } c}} \int h \, \mathrm{d}(\mathfrak{m}_x - \mathfrak{m}_y).$$

Likewise, not the dual Monge-Kantorovic duality.

Weak limit preservation

Weak pullbacks are not preserved.

Projective limits: Weak preservation can not be assumed in general, but without additional assumption, even a limit along

$$(X1, \mathcal{A}1) \twoheadleftarrow (X2, \mathcal{A}2) \twoheadleftarrow \ldots$$

does not exit (Andersen and Jessen 1948).

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What we want

A restriction of the Giry monad still comprising everything necessary for application (e.g. perfect measures on countably fibered spaces).

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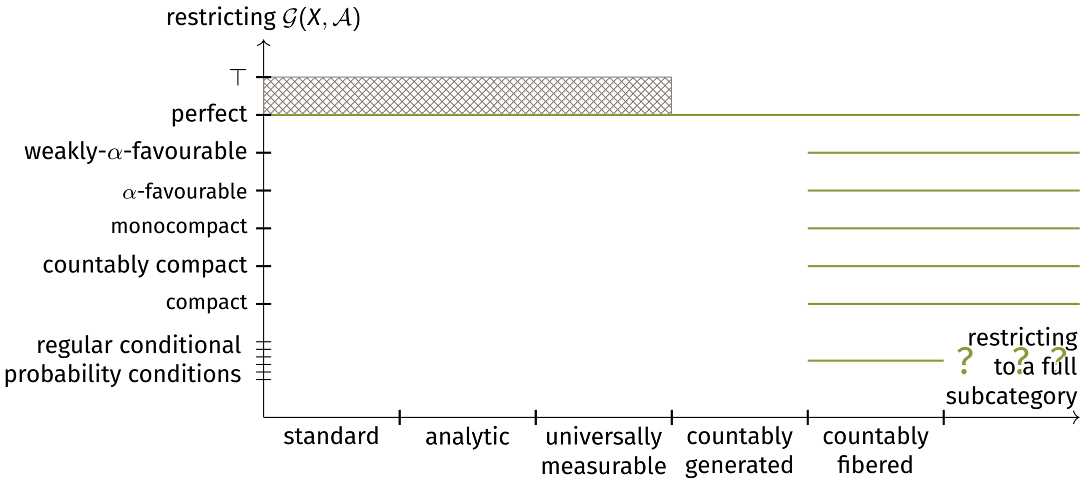
Idea: Restricting Giry Monad on Meas



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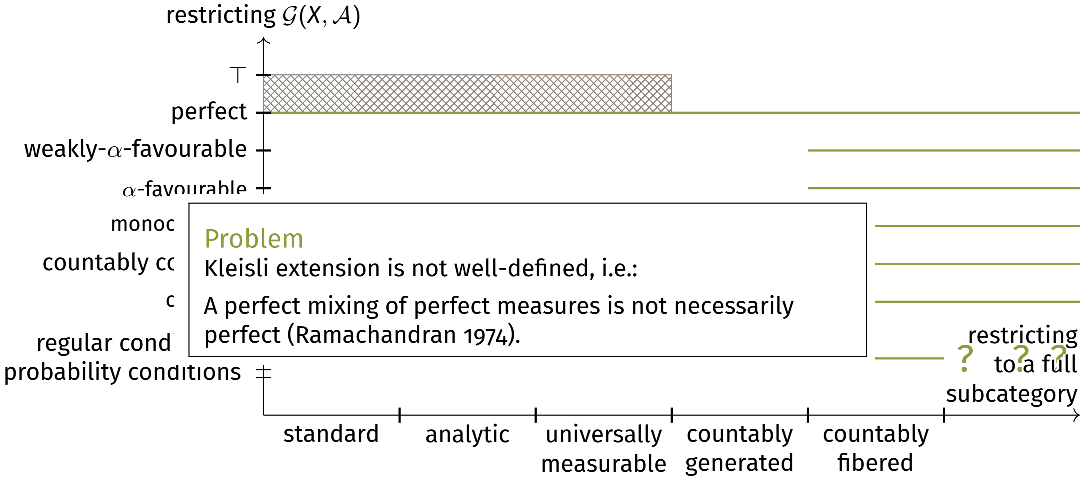

Problem

Kleisli extension is not well-defined, i.e.:

A perfect mixing of perfect measures is not necessarily perfect (Ramachandran 1974).

What can be done

Remaining structure

We still have

- an endofunctor $\mathcal{G}_{perf} : \mathbf{Meas} \to \mathbf{Meas}$
  $$(X, \mathcal{A}) \mapsto \left\{ \begin{matrix} \text{perf. prob. m.} \\ \text{on } (X, \mathcal{A}) \end{matrix} \right\}$$
- with a unit
  ($\rightsquigarrow$ well-pointed endofunctor).

Alternatively, $\mathcal{G}_{\text{perf}}$ can be viewed as a *relative monad* on the identity functor to the category of partial measurable maps.

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Weak pullback-preservation

If we restrict further to an endofunctor $\mathcal{G}_{\text{rcpp}}$

$$(X, \mathcal{A}) \mapsto \left\{ \begin{array}{l} \text{perf. prob. m.} \\ \quad \text{on } (X, \mathcal{A}) \text{ with} \\ \quad \text{subfield rcpp (*)} \end{array} \right\}$$

for $(X, \mathcal{A})$ countably fibered, $\mathcal{G}_{\text{rcpp}}$ preserves weak pullbacks.

(*) regular conditional probability condition

Projective limits

Exist under optimal condition for countably fibered spaces (Musiał 1980).

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Clue for real-world application

When viewed as a relative monad one could escape in the following way:

1. postulate that the extensive quantities you want to model by probability measures are perfect.
2. Do some mathematical arguments, resulting in the desired statement provided that mixing goes well
3. using the postulate to say that, as the resulting probability measures exist, they must be perfect.

Conceptually not satisfying.

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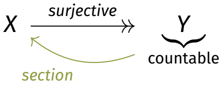
In ZF + CC + AD (with a grain of salt)

Restricting Axiom of Choice

When restricting to the Axiom of Countable choice (CC)



theory of integration and lot more still goes through.
On the other hand, ZF + CC is consistent with AD, the Axiom of Determinancy.

Measure theoretic consequences of AD

All subsets of $\mathbb{R}$ are Lebesgue measurable.

All prob. m.'s are perfect.

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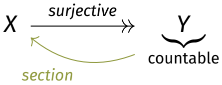

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The Giry monad in ZF + CC + AD

As $\mathcal{G}_{\text{perf}} = \mathcal{G}$, it is a monad.

Restricting to $\mathcal{G}_{rcpp}$ we obtain again the weak limit preservation properties from above.

The objects $\mathcal{G}(X, \mathcal{A})$

$$\mathcal{G}(\text{countably generated}) = \text{standard}$$
$$\mathcal{G}(\text{countably fibered}) = \text{analytic}$$
$$\mathcal{G}(\text{arbitray}) = \text{smooth}$$

Smooth space generalise analytic spaces going back to Falkner (1981).
Many constructions actually work for smooth spaces, e.g. behavioral distance of Markov decision systems (Beohar, L., Kupke 2025).

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