Upgrading equivalences in a weak ω-category to coherent ones

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me

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Upgrading an equivalence to an adjoint one

Well known (2-categorical case):

Given $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$, $\alpha : id_{\mathcal{C}} \cong GF$ and $\beta : FG \cong id_{\mathcal{D}}$.

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$$(F, G, \alpha, \beta')$$

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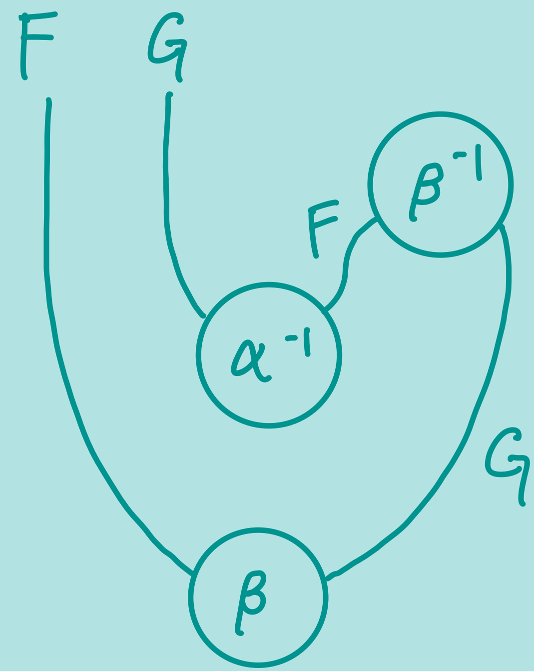

Upgrading an equivalence to an adjoint one

Well known (2-categorical case):

Given $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $\alpha : \mathrm{id}_{\mathcal{C}} \xrightarrow{\simeq} GF$ and $\beta : FG \xrightarrow{\simeq} \mathrm{id}_{\mathcal{D}}$,

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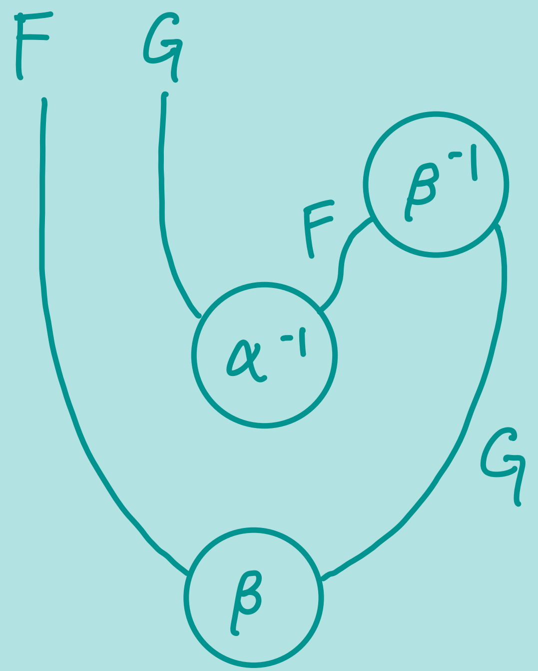
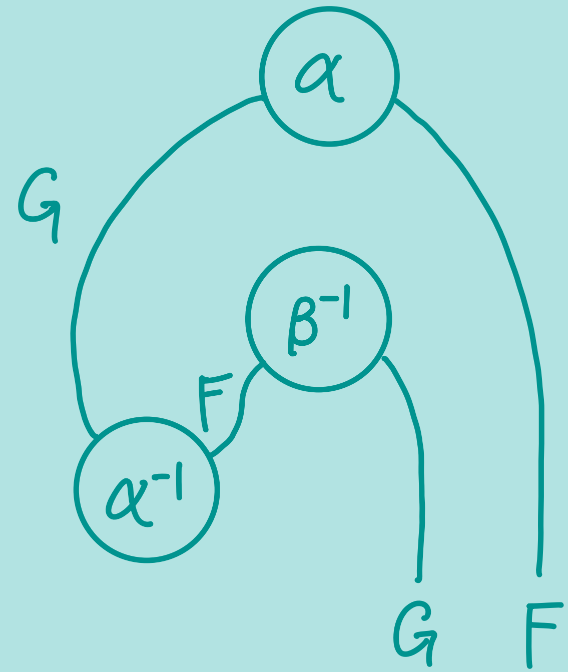
(F, G, α, β') or dually (F, G, α', β).

$\beta' =$ 

$\alpha' =$ 

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In what sense are adjoint equivalences better than mere equivalences?

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Example:

Transferring a monoidal structure along an equivalence of categories
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}.$$

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Yes, the free 2-category containing an adjoint equivalence is biequivalent to a point.

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Specifying an adjoint equivalence in $\mathcal{K}$ $\quad (\mathbb{A}\mathrm{dj}\,\mathbb{E}\mathrm{q} \longrightarrow \mathcal{K})$

$\Updownarrow$ essentially the same

Specifying a single object in $\mathcal{K}$ $\quad (* \longrightarrow \mathcal{K})$

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Theorem (Fujii - Hoshino - M.):

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- E is suitably equivalent to a point $\rightsquigarrow$ E is "coherent"
- any "equivalence" in a weak ω-category X can be detected by an ω-functor $E \longrightarrow X$.

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  (This part is deliberately paraphrased in a misleading way.)

- any "equivalence" in a weak ω-category X can be detected by an ω-functor $E \longrightarrow X$.

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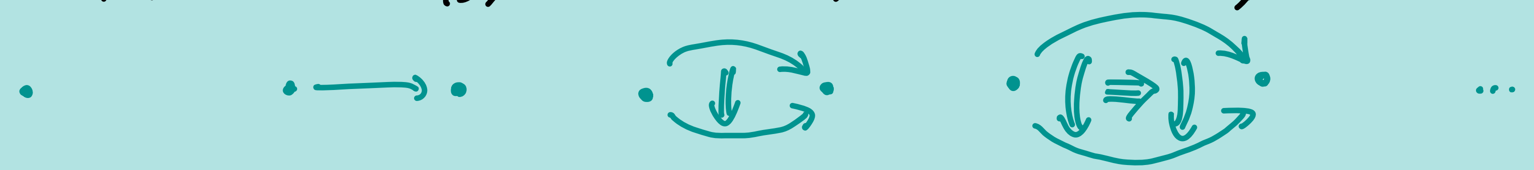
Weak ω-categories

Informal definition:

A weak ω-category consists of

0-cells, 1-cells, 2-cells, 3-cells, ...



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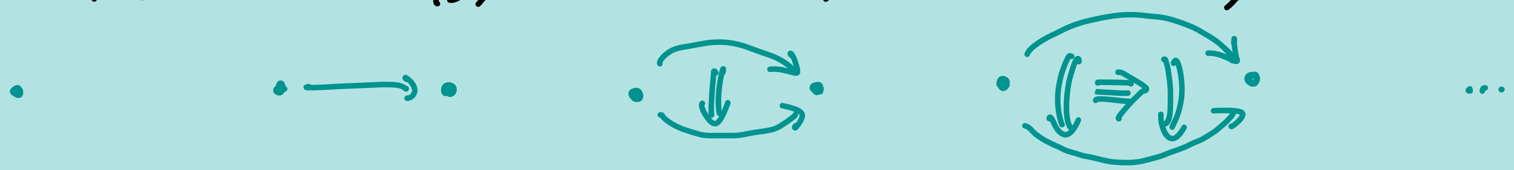

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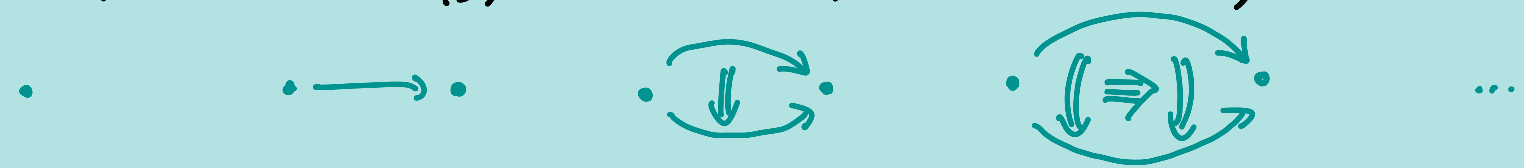

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Slightly more formal definition (Batanin, Leinster):

A weak ω-category is an EM-algebra for the "universal weakening" of the monad for strict ω-categories.

Coinductive equivalences

Definition:

An n-cell $f : x \to y$ in a weak ω-category is an equivalence if there exist

- an n-cell $g : y \to x$,
- an equivalence $(n+1)$-cell $u : 1_x \to gf$, and
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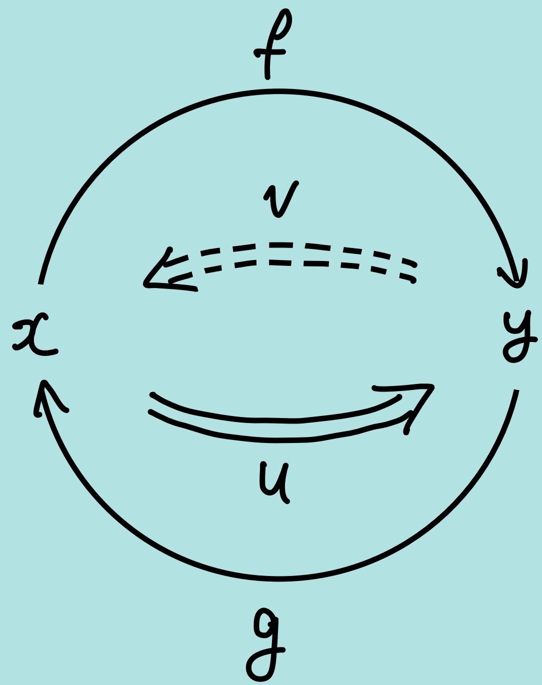
Topological intuition

The "free equivalence" is made of :

$$\begin{tikzcd}[column sep=huge] x \arrow[r, bend left=80, "f"] \arrow[r, phantom, shift left=2, "\overset{v}{\Leftarrow}"] \arrow[r, phantom, shift right=2, "\underset{u}{\Rightarrow}"] & y \arrow[l, bend left=80, "g"] \end{tikzcd}$$

Topological intuition

The "free equivalence" is not coherent because it is made of spheres :



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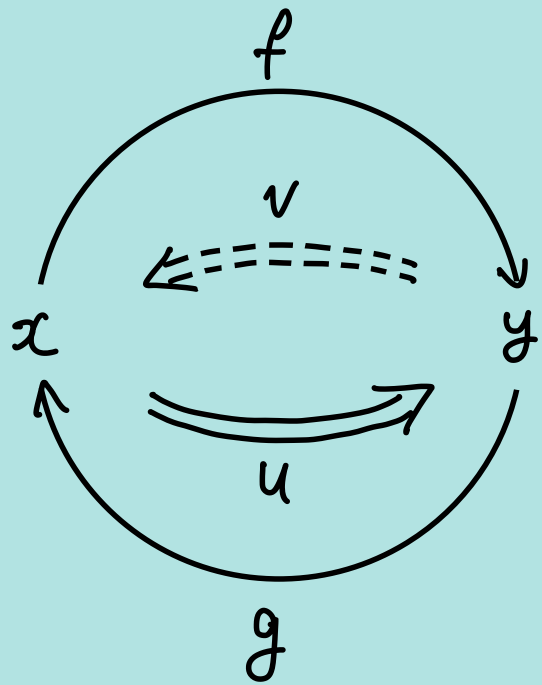

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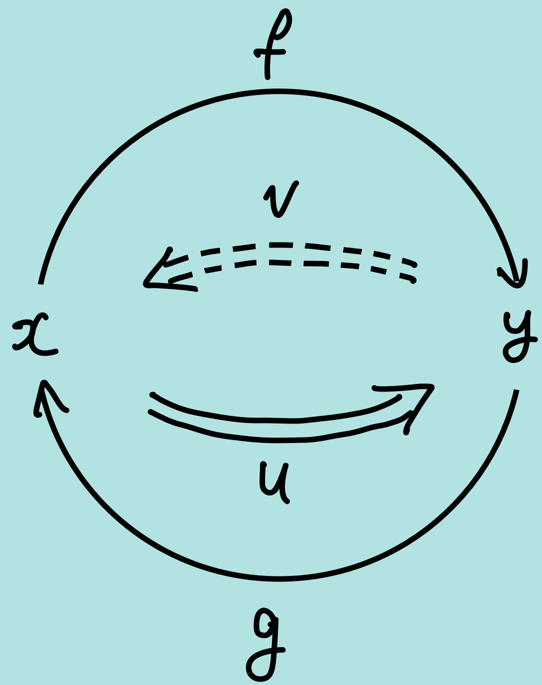


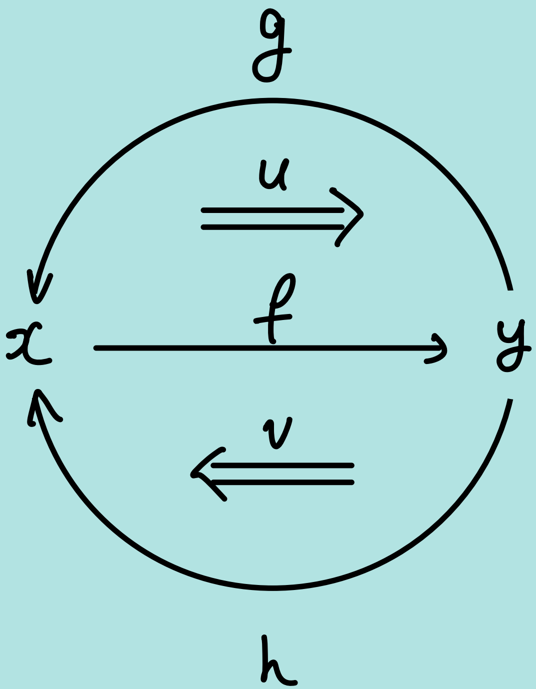
In order to make it equivalent to a point, we need to

- cut it open to obtain a disk, or
- fill it to obtain a ball.

Two approaches (cf. [R])

<u>Cutting open the sphere:</u>

Ask for separate left and right inverses.



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$$
\begin{tikzcd}
 & \overset{u}{\Rightarrow} & \\
x \arrow[rr, "f"] & & y \arrow[ll, bend right=80, "g"'] \arrow[ll, bend left=80, "h"] \\
 & \overset{v}{\Leftarrow} &
\end{tikzcd}
$$

Filling the sphere:

Ask for one of the triangle identities.

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=4ex, "\overset{u}{\Nearrow}"] & y \arrow[d, "g"'] \arrow[dr, "1"] \arrow[dr, phantom, shift right=4ex, "\overset{v'}{\Nearrow}"] & \\
 & x \arrow[r, "f"'] & y
\end{tikzcd}
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$\Downarrow$ equivalence

1_f

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In the "cutting open" approach, the free coherent equivalence looks like:

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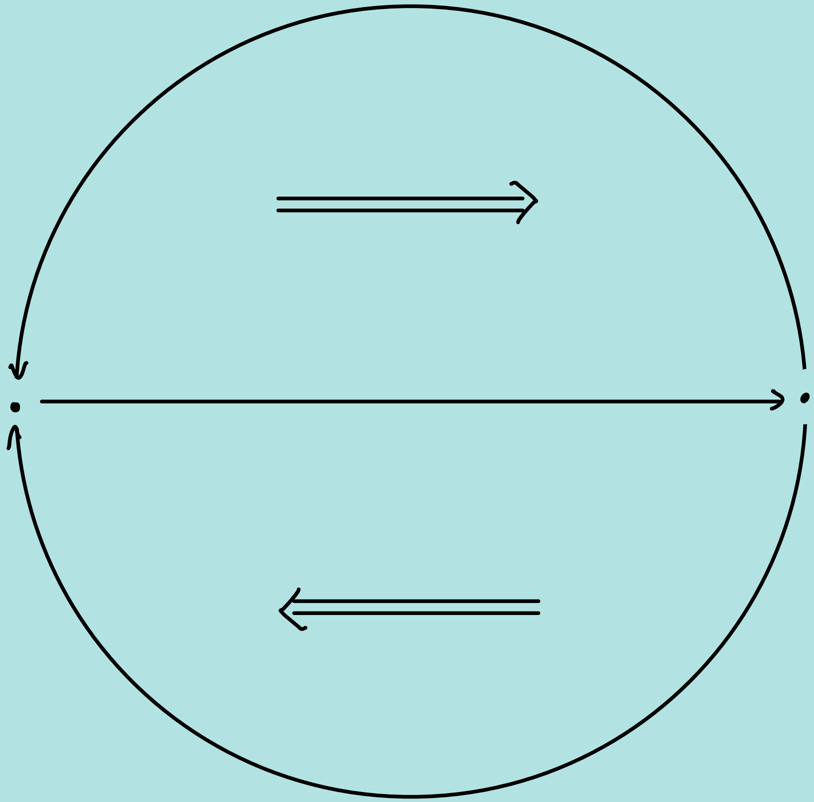
$E =$ 

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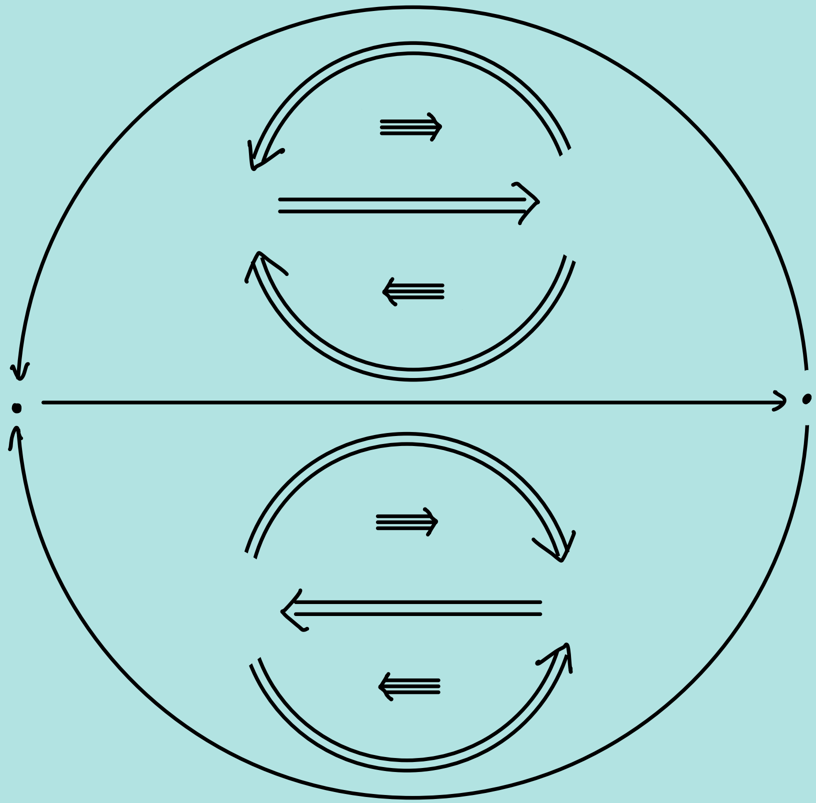
$E =$ 

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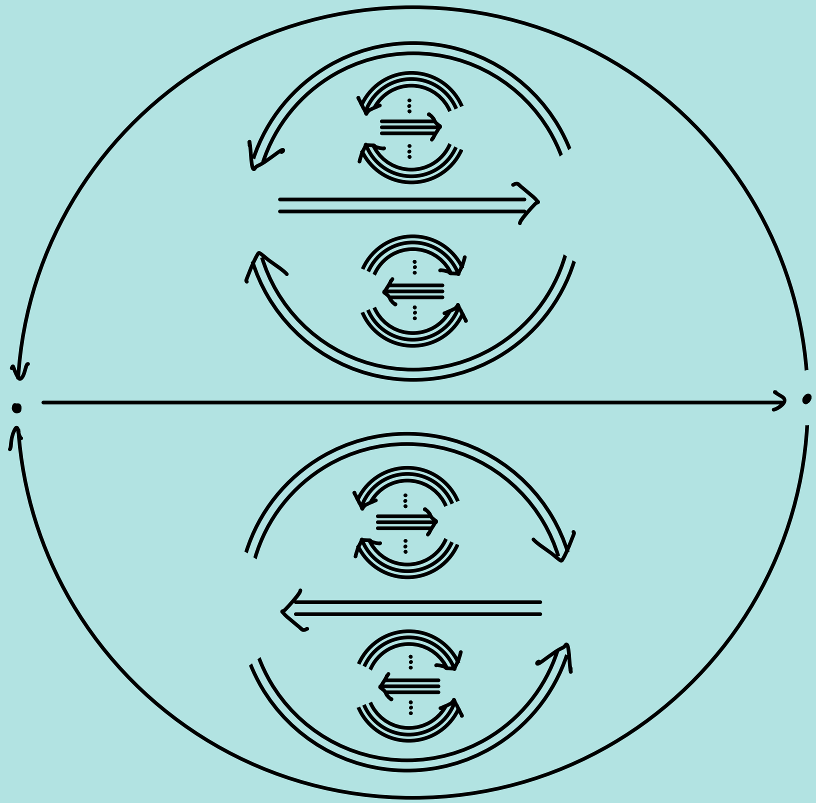
E = 

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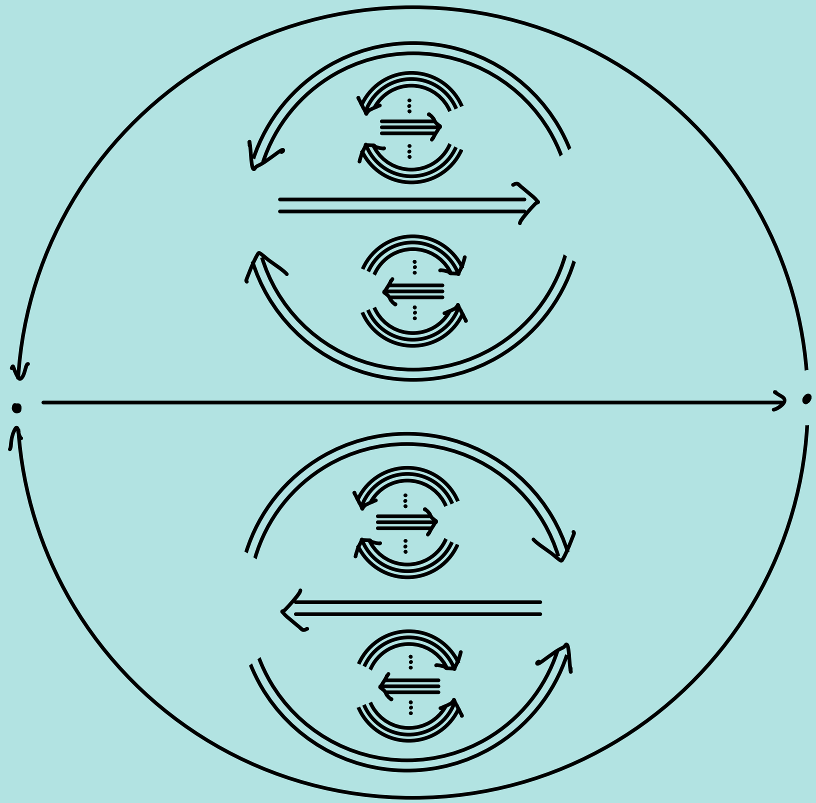
$E \quad =$ 

Applying $\mathrm{Wk}\text{-}\omega\text{-}\mathrm{Cat} \underset{\perp}{\longrightarrow} \mathrm{Str}\text{-}\omega\text{-}\mathrm{Cat}$ to E yields $\widehat{\omega E}$ in [OR].

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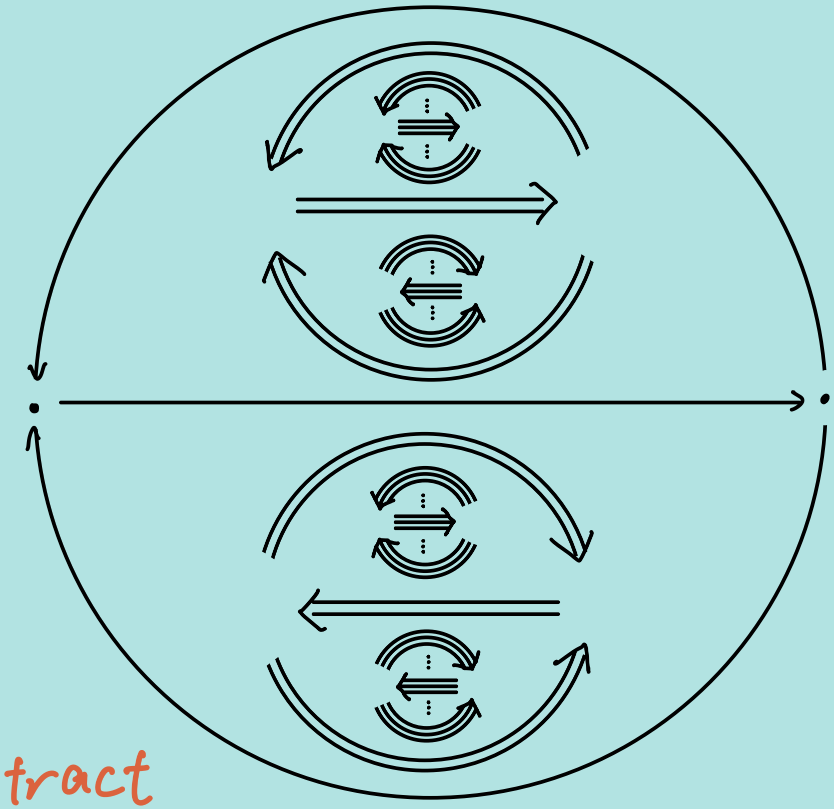
$E \; =$ 

missing in the abstract

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Main result

<u>Definition:</u>

An ω-functor $F : X \longrightarrow Y$ between weak ω-categories is an ω-equifibration if

$\forall x \in X_n \quad \forall \text{eq. } Fx \xrightarrow{f} y \in Y_{n+1} \quad \exists \text{eq. } x \xrightarrow{\bar{f}} \bar{y} \quad \text{s.t. } F\bar{f} = f$

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Theorem (Fujii - Hoshino - M.):

$F : X \longrightarrow Y$ is an ω-equifibration if and only if F has the RLP against $\{ \underbrace{\Sigma^n}_{\text{suspension}} (* \longrightarrow \underbrace{E}_{\text{can use either version}}) \mid n \geq 0 \}$.

Relation to homotopy theory of strict ω-categories

Theorem (Fujii - Hoshino - M.):

Applying $$\begin{tikzcd} \mathrm{Wk}\text{-}\omega\text{-}\mathrm{Cat} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Str}\text{-}\omega\text{-}\mathrm{Cat} \arrow[l, bend left] \end{tikzcd}$$ to

$$\{ \Sigma^n (* \longrightarrow E) \mid n \geq 0 \}$$

yields a generating set of trivial cofibrations in the folk model structure on $\mathrm{Str}\text{-}\omega\text{-}\mathrm{Cat}$.

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Consequently, the folk fibrations are precisely the ω-equifibrations between strict ω-categories.

References

[L] Stephen Lack, "A Quillen model structure for bicategories"
DOI: 10.1007/s10977-004-6757-9

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