Weak action representability and categories of algebras

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Action representable categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X)$ is representable.

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- Prototype examples: **Grp**, **Lie** and any abelian category.
- In **Grp**, the actor of X is $\mathrm{Aut}(X)$.
- In **Lie**, the actor of X is $\mathrm{Der}(X)$.
- In any abelian category, the actor of X is 0.

Weakly action representable categories

Definition (G. Janelidze, 2022)

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An object T as above is called *weak representing object* of X, the pair (T, τ) is called *weak representation* of $\mathrm{SplExt}(-, X)$ and $(\varphi \colon B \to T) \in \mathrm{Im}(\tau_B)$ is called *acting morphism*.

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- Examples: **Assoc**, **Leib** and any action representable category.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden, C. Vienne, 2025)

Let $\mathcal{V}$ be an action accessible variety of non-associative algebras, let $\mathbf{PAlg}$ *be the category of partial algebras and let* $U\colon \mathcal{V} \to \mathbf{PAlg}$ *be the forgetful functor.*

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Let $\mathcal{C}$ be an action representable category and let $\mathcal{X}$ be a Birkhoff subcategory of $\mathcal{C}$. Suppose there exist two monomorphisms $m\colon S \rightarrowtail B$, $m'\colon S \rightarrowtail B'$ in $\mathcal{X}$, two monomorphisms $u\colon B \rightarrowtail D$, $u'\colon B' \rightarrowtail D$ in $\mathcal{C}$, an object X of $\mathcal{X}$ and a monomorphism $v\colon D \rightarrowtail [X]$ such that:

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Then, the category $\mathcal{X}$ is not weakly action representable.

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- **Open question:** is the variety $\mathbf{Nil}_2(\mathbf{Grp})$ weakly action representable?

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A variety of non-associative algebras $\mathcal{V}$ is *unit-closed* if for any algebra X of $\mathcal{V}$, the algebra $\tilde{X}$ obtained by adjoining to X the external unit 1, together with the identities $x \cdot 1 = 1 \cdot x = x$, is still an object of $\mathcal{V}$.

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Remark

The subcategory $\mathcal{V}_1$ of unitary algebras of $\mathcal{V}$ is an *ideally exact category*.

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Definition (G. Janelidze, 2024)

Let $\mathcal{C}$ be a category with pullbacks, with initial object 0 and terminal object 1. $\mathcal{C}$ is *ideally exact* if it is Barr exact, protomodular, has finite coproducts and the unique morphism $0 \to 1$ is a regular epimorphism.

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Remark (G. Janelidze, 2024)

There is a semi-abelian category $\mathcal{V}$ and a monadic adjunction with cartesian unit

$$\mathcal{C} \underset{U}{\overset{F}{\leftrightarrows}} \mathcal{V}, \quad F \dashv U \ (\perp).$$

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- One may take $\mathcal{V} = (\mathcal{C} \downarrow 0)$ with U and F defined in the obvious way.

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- One may take $\mathcal{V} = (\mathcal{C} \downarrow 0)$ with U and F defined in the obvious way.
- If $\mathcal{V}$ is a unit-closed variety of non-associative algebras, then the monadic adjunction with cartesian unit is given by

$$\mathcal{V}_1 \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{V}, \quad (F \dashv U)$$

where U forgets the unit and $F(X) = \mathbb{F} \ltimes X$, for any X in $\mathcal{V}$.

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Theorem (G. Janelidze, 2024)

*The ideally exact categories **Ring** and **CRing** are action representable.*

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Theorem (G. Janelidze, 2024)

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Theorem (M. M., F. Piazza, 2025)

The ideally exact categories $\mathbf{Assoc}_1$, $\mathbf{CAssoc}_1$, $\mathbf{Alt}_1$, $\mathbf{Pois}_1$, $\mathbf{CPois}_1$ *are action representable.*

Thank you!

- J. Brox, X. García-Martínez, M. Mancini, T. Van der Linden and C. Vienne, *Weak representability of actions of non-associative algebras*, Journal of Algebra **669** (2025), no. 18, 401–444.
- A. S. Cigoli, M. Mancini and G. Metere, *On the representability of actions of Leibniz algebras and Poisson algebras*, Proceedings of the Edinburgh Mathematical Society **66** (2023), no. 4, 998–1021.
- X. García-Martínez and M. Mancini, *Action accessible and weakly action representable varieties of algebras*, Bollettino dell'Unione Matematica Italiana (2025), accepted for publication, preprint arXiv:2503.17326, 2025.
- M. Mancini and F. Piazza, *Action representability in categories of unitary algebras* (2025), submitted, preprint arXiv:2503.04488.