Bicategories of lax fractions

Graham Manuell
graham@manuell.me
University of Stellenbosch

Joint work with Lurdes Sousa

CT2025, Brno

Classical categories of fractions

We sometimes want to freely adjoin inverses of certain morphisms to a category.

If $\mathcal{X}$ is a category and Σ is a class of morphisms in $\mathcal{X}$ there is a universal category $\mathcal{X}[\Sigma^{-1}]$ equipped with a functor $\mathcal{X} \to \mathcal{X}[\Sigma^{-1}]$ which sends morphisms in Σ to isomorphisms.

The construction of the category $\mathcal{X}[\Sigma^{-1}]$ is somewhat involved, but in good cases it can be greatly simplified: when Σ admits a calculus fractions the morphisms of $\mathcal{X}[\Sigma^{-1}]$ are given by equivalence classes of certain (co)spans.

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Calculi of fractions

The class Σ admits a calculus of (left) fractions if

- Σ is a wide subcategory of $\mathcal{X}$.
- Given a span $\bullet \xleftarrow{f} \bullet \xrightarrow{s} \bullet$ with $s \in \Sigma$ there is a commutative square

$$\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \bullet & \xrightarrow[s']{} & \bullet \end{array}$$

  with $s' \in \Sigma$.
- Given a diagram $\bullet \xrightarrow{r} \bullet \overset{f}{\underset{g}{\rightrightarrows}} \bullet$ with $fr = gr$ and $r \in \Sigma$, there is morphism $s \in \Sigma$ that coequalises f and g.

The resulting category of fractions

If Σ admits a category of fractions, then the category of fractions $\mathcal{X}[\Sigma^{-1}]$ has

- an object for each object in $\mathcal{X}$,
- morphisms $A \to B$ given by equivalence classes of cospans $A \xrightarrow{f} I \xleftarrow{r} B$ where $r \in \Sigma$.

The cospans $A \xrightarrow{f} I \xleftarrow{r} B$ and $A \xrightarrow{g} J \xleftarrow{s} B$ are equivalent if there is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & I & \xleftarrow{r} & B \\ \| & & \downarrow{\scriptstyle x_1} & & \| \\ \| & & X & \xleftarrow{x_3} & B \\ \| & & \uparrow{\scriptstyle x_2} & & \| \\ A & \xrightarrow[g]{} & J & \xleftarrow[s]{} & B \end{array}$$

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where $x_3 \in \Sigma$.

The resulting category of fractions — composition

To compose $A \xrightarrow{f} I \xleftarrow{r} B$ and $B \xrightarrow{g} J \xleftarrow{s} C$ we find a composite cospan

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & I & \xleftarrow{r} & B & & \\
 & & \big\downarrow{\scriptstyle g'} & & \big\downarrow{\scriptstyle g} & & \\
 & & \bullet & \xleftarrow[r']{} & J & \xleftarrow[s]{} & C
\end{array}
$$

where the square is from the 2^{nd} axiom of a calculus of left fractions.

Bicategories of fractions

Pronk extended the notion of calculus of fractions to the 2-dimensional setting.

Given a bicategory $\mathcal{X}$, we wish to freely turn the 1-morphisms in a given class Σ into *equivalences*. The axioms for a 2-dimensional calculus of fractions are a reasonable extension of the 1-dimensional ones we considered before.

This construction is particularly useful for defining the correct morphisms for internal categories.

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The case of monoidal categories

For example, internal categories in the category of monoids **Mon** are (strict) monoidal categories and the internal functors are strict monoidal functors.

Formally inverting fully faithful internal functors which are regular epimorphisms on objects results in what is essentially the bicategory of (strict) monoidal categories and *strong* monoidal functors.

We might ask if there is a way to describe *lax* monoidal functors in a similar way.

In fact, we can do this by forcing certain internal functors to become *adjoints*.

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Bicategories of lax fractions

Rather than turning certain 1-morphisms into equivalences, we want to be able to turn them into *left adjoint right inverses* (or left adjoint *left* inverses, etc).

It is useful to also be able to specify which squares are to become Beck–Chevalley (or BC) squares. Recall that if r, s are left adjoints and δ is invertible we say that

$$\begin{array}{ccc} \bullet & \xrightarrow{r} & \bullet \\ {\scriptstyle f}\downarrow & \overset{\delta}{\nearrow} & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[s]{} & \bullet \end{array}$$

is a BC square if the mate $\bar{\delta}$ in

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The universal property for lax fractions

Let $\mathcal{X}$ be a 2-category and let Σ be a collection of pseudocommutative squares in $\mathcal{X}$.

$$\begin{array}{ccc} \bullet & \xrightarrow{r} & \bullet \\ {\scriptstyle f}\big\downarrow & \overset{\delta}{\nearrow} & \big\downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[s]{} & \bullet \end{array}$$

We call these squares Σ-squares and the horizontal morphisms in such squares Σ-morphisms.

The bicategory of (left) lax fractions $\mathcal{X}[\Sigma_*]$ comes equipped with a pseudofunctor $P_\Sigma \colon \mathcal{X} \to \mathcal{X}[\Sigma_*]$ which is universal amongst pseudofunctors which send Σ-morphisms to left adjoint right inverses and Σ-squares to BC squares.

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A lax calculus of fractions

Let $\mathcal{X}$ be a 2-category and let Σ be a sub-double-category of the double category $\mathrm{Sq}(\mathcal{X})$ of quintets. In other words, Σ is a collection of squares in $\mathcal{X}$ commuting up to isomorphism that is closed under horizontal and vertical composition.

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Identity. Every object $X \in \mathcal{X}$ lies in Σ, and for every Σ-morphism $s\colon X \to Y$ we have the Σ-square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle 1_X}\downarrow & \Sigma^{\mathrm{id}} & \downarrow{\scriptstyle s} \\ X & \xrightarrow[s]{} & Y. \end{array}$$

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Vertical Repletion. If $r \colon X \to Y$ is a Σ-morphism and $\delta \colon r \Rightarrow s$ is an invertible 2-cell, then s is Σ-morphism and we have the Σ-square

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle 1_X}\downarrow & \Sigma^{\delta} & \downarrow{\scriptstyle 1_Y} \\ X & \xrightarrow[r]{} & Y. \end{array}$$

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Horizontal Repletion. For every pair of morphisms $f, g \colon X \to Y$ and every invertible 2-cell $\gamma \colon f \Rightarrow g$, we have the Σ-square

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle f}\downarrow & \Sigma^{\gamma} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[1_Y]{} & Y. \end{array}$$

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Square. For every Σ-morphism s and span $\bullet \xleftarrow{f} \bullet \xrightarrow{s} \bullet$ there is a Σ-square

$$\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ {\scriptstyle f}\downarrow & \Sigma^{\alpha} & \downarrow{\scriptstyle f'} \\ \bullet & \xrightarrow[s']{} & \bullet \end{array}.$$

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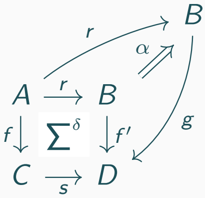
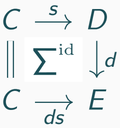
Equi-insertion. For every Σ-square and every 2-cell $\alpha\colon f'r \Rightarrow gr$ as in the diagram below on the left, there is a Σ-square as on the right and a 2-cell $\alpha'\colon df' \Rightarrow dg$ such that $d\alpha = \alpha' r$.





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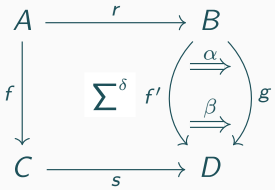
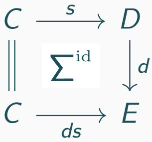
Equification. For every Σ-square and pair of 2-cells, as in the diagram on the left, with $\alpha r = \beta r$, there is a Σ-square as on the right such that $d\alpha = d\beta$.





The bicategory of lax fractions

If $(\mathcal{X}, \Sigma)$ admits a calculus of lax fractions we can form the bicategory $\mathcal{X}[\Sigma_*]$ as follows.

- The objects are simply the objects of $\mathcal{X}$.
- The 1-cells $(f, I, r)\colon X \to Y$ are cospans $X \xrightarrow{f} I \xleftarrow{r} Y$.
- The 2-cells $\overline{\alpha}\colon (f, I, r) \to (g, J, s)$ are given by equivalence classes of diagrams of the form:

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A & \xrightarrow{f} & I & \xleftarrow{r} & B \\
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\Vert & \overset{\alpha}{\Swarrow} & X & \xleftarrow{x_3} & B \\
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Composition and coherence

Composition of 1-cells is similar to the 1d case. We omit the descriptions of vertical and horizontal composition of 2-cells.

To deal with the associators and issues of coherence, we defined a class of 'simple' 2-cells and proved that these compose in a particularly simple way.

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Ω 2-cells

Given two different Σ-squares with the same top-left span,

$$\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ {\scriptstyle f}\downarrow & \Sigma^{\alpha_1} & \downarrow{\scriptstyle f_1'} \\ \bullet & \xrightarrow[s_1']{} & \bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ {\scriptstyle f}\downarrow & \Sigma^{\alpha_2} & \downarrow{\scriptstyle f_2'} \\ \bullet & \xrightarrow[s_2']{} & \bullet \end{array},$$

our axioms allow us to construct a 2-cell in $\mathcal{X}[\Sigma_*]$ that relates the bottom-right cospans:

$$\begin{array}{ccccc} A & \xrightarrow{f_1'} & I & \xleftarrow{r_1'} & B \\ \| & & \downarrow & \Sigma & \| \\ \| & \overset{\Omega}{\swarrow} & X & \longleftarrow & B \\ \| & & \uparrow & \Sigma & \| \\ A & \xrightarrow[f_2']{} & J & \xleftarrow[r_2']{} & B. \end{array}$$

This 2-cell is canonical in the sense that any choices made in its construction do not change its equivalence class.

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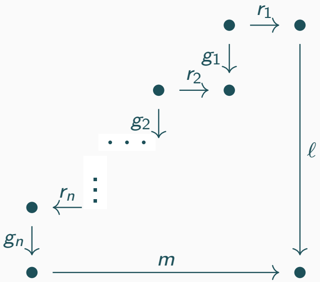
Σ-paths

Now given two staircase diagrams of the form



with the same top-left border, each tiled with Σ-squares, we can obtain a 2-cell between the bottom-right borders by replacing similar Σ-squares and composing the resulting Ω 2-cells.

However, it is now a rather difficult theorem to show that the order in which we make these swaps (at least for the cases we need) does not affect the resulting composite 2-cell.

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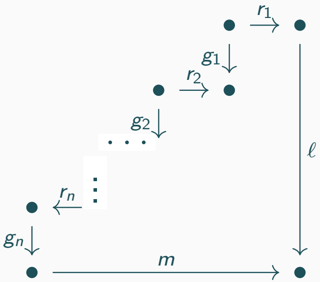

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An application

It is possible to describe the Kleisli category of a lax idempotent pseudomonad in terms of lax fractions.

In particular, this (or rather its dual) allows us to recover our bicategory of (strict) monoidal categories and lax monoidal functors by taking:

- $\mathcal{X} = \mathbf{Cat}(\mathbf{Mon})$
- The Σ-morphisms to be the strict monoidal functors whose underlying functors have fully faithful right adjoints,
- The Σ-squares to be the BC squares for the underlying functors,

and finding the bicategory of *right* lax fractions.

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