The cohomology of semi abelian variety are small

Mattenet, Van der Linden, Jungers

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Presentation

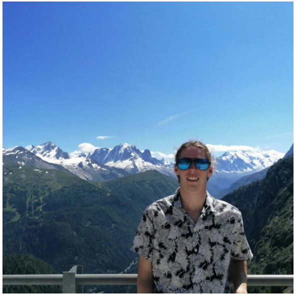

Figure: Sébastien

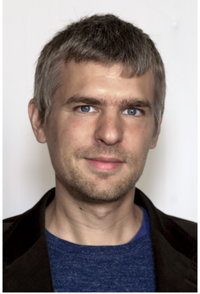

Figure: Tim Van der Linden

Figure: Raphaël Jungers

This work can be found on arxiv as [Mattenet et al., 2025] and will be published soon in Theory and Application of Category.

Overview

1. Introduction
2. Ext^n reduce to Ext^1
3. Varieties of algebras

Co-homology categories

Exact sequences between K and Q

$$0 \to K \to X_n \to \cdots \to X_1 \to Q \to 0$$

and morphisms

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f_{n+1}"] \arrow[d, dashed] & X_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"] & X_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & \cdots \arrow[r] & X_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & X_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & Q \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & L \arrow[r, "g_{n+1}"'] & Y_n \arrow[r, "g_n"'] & Y_{n-1} \arrow[r] & \cdots \arrow[r] & Y_2 \arrow[r, "g_2"'] & Y_1 \arrow[r, "g_1"'] & R \arrow[r] & 0
\end{tikzcd}$$

Form a category $\mathrm{Ext}^n(Q, K)$

Co-homology categories

Exact sequences between K and Q

$$0 \to K \to X_n \to \cdots \to X_1 \to Q \to 0$$

and morphisms

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0 \arrow[r] & L \arrow[r, "g_{n+1}"'] & Y_n \arrow[r, "g_n"'] & Y_{n-1} \arrow[r] & \cdots \arrow[r] & Y_2 \arrow[r, "g_2"'] & Y_1 \arrow[r, "g_1"'] & R \arrow[r] & 0
\end{tikzcd}$$

Form a category $\mathrm{Ext}^n(Q, K)$

Research question

Question

IF C has enough to define exact sequences, is $\mathrm{Ext}^n(Q, K)$ small?

Previous work

In the abelian case

1. Model Categories
2. Localisation
3. Derived categories
4. Categories of fractions
5. Cohomology as interpretation of something else

Semi-abelian case

1. unclear works in general
2. ??
3. does not apply
4. ??
5. still need to know cohomology from elsewhere

Previous work

In the abelian case	Semi-abelian case
1. Model Categories	1. unclear works in general
2. Localisation	2. ??
3. Derived categories	3. does not apply
4. Categories of fractions	4. ??
5. Cohomology as interpretation of something else	5. still need to know cohomology from elsewhere

Plan

Almost an induction

1. Show that under nice case we can reduce $\mathrm{Ext}^n(Q, K)$ to the case $\mathrm{Ext}^1(Q, K)$
2. Show that variety of algebra are nice and have small $\mathrm{Ext}^1(Q, K)$.

The syzygy argument

A syzygy of an object Q is a short exact sequence

$$0 \longrightarrow \Omega(Q) \xrightarrow{w} P \xrightarrow{p} Q \longrightarrow 0$$

where the middle object P is normal-projective, meaning we have
$$\begin{tikzcd} & X \arrow[d] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"'] & Y \end{tikzcd} .$$

$$\begin{tikzcd}
& & & & I_n \arrow[dr, "m_n"] & & & & \cdots \arrow[dr, "m_2"] \\
0 \arrow[r] & K \arrow[rr, "f_{n+1}"] \arrow[dr, equal] & & X_n \arrow[ur, "e_n"] \arrow[rr, "f_n"'] & & X_{n-1} \arrow[rr, "f_{n-1}"] \arrow[dr, "e_{n-1}"'] & & X_{n-2} \arrow[r] \arrow[ur] & \cdots \arrow[r] & X_1 \arrow[rr, "f_1"] \arrow[dr, "e_1"'] & & Q \arrow[r] & 0 \\
& & I_{n+1} \arrow[ur, "m_{n+1}"'] & & & & I_{n-1} \arrow[ur, "m_{n-1}"'] & & & & I_1 \arrow[ur, equal]
\end{tikzcd}$$

The syzygy argument

A syzygy of an object Q is a short exact sequence

$$0 \longrightarrow \Omega(Q) \xrightarrow{\ w\ } P \xrightarrow{\ p\ } Q \longrightarrow 0$$

where the middle object P is normal-projective, meaning we have
$$\begin{tikzcd}
 & X \arrow[d, two heads] \\
P \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"'] & Y
\end{tikzcd}.$$

$$\begin{tikzcd}
 & & & & I_n \arrow[dr, bend left, "m_n"] & & & & \cdots \arrow[dr, bend left, "m_2"] & & & & \\
0 \arrow[r] & K \arrow[rr, "f_{n+1}"] \arrow[dr, equal, bend right] & & X_n \arrow[rr, "f_n"'] \arrow[ur, bend left, "e_n"] & & X_{n-1} \arrow[rr, "f_{n-1}"] \arrow[dr, bend right, "e_{n-1}"'] & & X_{n-2} \arrow[r] \arrow[ur, bend left] & \cdots \arrow[r] & X_1 \arrow[rr, "f_1"] \arrow[dr, bend right, "e_1"'] & & Q \arrow[r] & 0 \\
 & & I_{n+1} \arrow[ur, bend right, "m_{n+1}"'] & & & & I_{n-1} \arrow[ur, bend right, "m_{n-1}"'] & & & & I_1 \arrow[ur, equal, bend right] & &
\end{tikzcd}$$

Assumption and goals

Assumption

C is semi-abelian

C has enough projective

Question

Is $\mathrm{Ext}^n(Q, K)$ small?

The syzygy argument

$$
\begin{tikzcd}
 & & & & & & & 0 \arrow[r] & \Omega(Q) \arrow[r, dotted] \arrow[dr, bend left, "w"] \arrow[dddd, dotted, "\overline{\alpha}_1"] & 0 & & \\
0 \arrow[r, dotted] & K \arrow[r, dotted] \arrow[dd, dotted, equal] & Y_n \arrow[r, dotted, "g_n"] \arrow[dd, dotted, equal, "\alpha_n"'] & Y_{n-1} \arrow[r, dotted] \arrow[dd, dotted, equal, "\alpha_{n-1}"'] & \cdots \arrow[r, dotted] & Y_2 \arrow[rr, dotted, "g_2"] \arrow[dd, dotted, equal, "\alpha_3"'] \arrow[dr, dotted, "e_3"] & & Y_1 \arrow[ur, dotted, "g_1"] \arrow[dd, dotted, "\alpha_2"] & & P \arrow[r, "p"] \arrow[dd, dotted, "\alpha_1"] & Q \arrow[r] \arrow[dd, equal] & 0 \\
 & & & & & & I_3 \arrow[ur, dotted, "l_2"] \arrow[dr, bend left, "m_3"] & & & & & \\
0 \arrow[r] & K \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"'] & X_n \arrow[r] & \cdots \arrow[r] & X_3 \arrow[rr, "f_3"'] \arrow[ur, bend left, "e_3"] & & X_2 \arrow[rr, "f_2"] \arrow[dr, bend right, "e_2"'] & & X_1 \arrow[r, "f_1"'] & Q \arrow[r] & 0 \\
 & & & & & & & & I_2 \arrow[ur, bend right, "m_2"'] & & &
\end{tikzcd}
$$

Figure: Syzygy and pullback

The syzygy argument

$$
\begin{tikzcd}
 & & & & & & & 0 \arrow[r] & \Omega(Q) \arrow[r, dotted] \arrow[dr, bend left, "w"] \arrow[dddd, dotted, "\overline{\alpha}_1"] & 0 & & \\
0 \arrow[r, dotted] & K \arrow[r, dotted] \arrow[dd, dotted, no head] & Y_n \arrow[r, dotted, "g_n"] \arrow[dd, dotted, no head, "\alpha_n"'] & Y_{n-1} \arrow[r, dotted] \arrow[dd, dotted, no head, "\alpha_{n-1}"'] & \cdots \arrow[r, dotted] & Y_2 \arrow[rr, dotted, "g_2"] \arrow[dr, dotted, "e_3"] \arrow[dd, dotted, no head, "\alpha_3"'] & & Y_1 \arrow[ur, dotted, bend left, "g_1"] \arrow[dd, dotted, "\alpha_2"] & & P \arrow[r, "\color{red}p"] \arrow[dd, dotted, "\color{blue}\alpha_1"] & Q \arrow[r] \arrow[dd, equal] & 0 \\
 & & & & & & I_3 \arrow[ur, dotted, "l_2"] \arrow[dr, bend left, "m_3"] & & & & & \\
0 \arrow[r] & K \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"'] & X_n \arrow[r] & \cdots \arrow[r] & X_3 \arrow[rr, "f_3"'] \arrow[ur, bend left, "e_3"] & & X_2 \arrow[rr, "f_2"] \arrow[dr, bend right, "e_2"'] & & X_1 \arrow[r, "\color{red}f_1"'] & Q \arrow[r] & 0 \\
 & & & & & & & & I_2 \arrow[ur, bend right, "m_2"'] & & &
\end{tikzcd}
$$

we use, we get, by universal property they are the same.

The syzygy argument

$$
\begin{tikzcd}
 & & & & & & & 0 \arrow[r] & \Omega(Q) \arrow[r, dotted] \arrow[dr, bend left, "w"] \arrow[dddd, dotted, "\overline{\alpha}_1"] & 0 & & \\
0 \arrow[r, dotted] & K \arrow[r, dotted] \arrow[dd, dotted, equal] & Y_n \arrow[r, dotted, "g_n"] \arrow[dd, dotted, "\alpha_n"'] & Y_{n-1} \arrow[r, dotted] \arrow[dd, dotted, "\alpha_{n-1}"'] & \cdots \arrow[r, dotted] & Y_2 \arrow[rr, dotted, "g_2"] \arrow[dr, dotted, "e_3"] \arrow[dd, dotted, "\alpha_3"'] & & Y_1 \arrow[ur, dotted, "g_1"] \arrow[dd, dotted, "\alpha_2"] & & P \arrow[r, "p"] \arrow[dd, dotted, "\alpha_1"] & Q \arrow[r] \arrow[dd, equal] & 0 \\
 & & & & & & I_3 \arrow[ur, dotted, "l_2"] \arrow[dr, bend left, "m_3"] & & & & & \\
0 \arrow[r] & K \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"'] & X_n \arrow[r] & \cdots \arrow[r] & X_3 \arrow[rr, "f_3"'] \arrow[ur, bend left, "e_3"] & & X_2 \arrow[rr, "f_2"] \arrow[dr, bend right, "e_2"'] & & X_1 \arrow[r, "f_1"'] & Q \arrow[r] & 0 \\
 & & & & & & & & I_2 \arrow[ur, bend right, "m_2"'] & & &
\end{tikzcd}
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The syzygy argument

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\begin{tikzcd}
 & & & & & & & 0 \arrow[r] & \Omega(Q) \arrow[r, dotted] \arrow[dr, bend left, "w"] \arrow[dddd, dotted, "\overline{\alpha}_1"] & 0 & & \\
0 \arrow[r, dotted] & K \arrow[r, dotted] \arrow[dd, equal, dotted] & Y_n \arrow[r, dotted, "g_n"] \arrow[dd, equal, dotted, "\alpha_n"'] & Y_{n-1} \arrow[r, dotted] \arrow[dd, equal, dotted, "\alpha_{n-1}"'] & \cdots \arrow[r, dotted] & Y_2 \arrow[rr, dotted, "g_2"] \arrow[dr, dotted, "e_3"] \arrow[dd, equal, dotted, "\alpha_3"'] & & Y_1 \arrow[ur, dotted, "g_1"] \arrow[dd, dotted, "\alpha_2"] & & P \arrow[r, "p"] \arrow[dd, dotted, "\alpha_1"] & Q \arrow[r] \arrow[dd, equal] & 0 \\
 & & & & & & I_3 \arrow[ur, dotted, "l_2"] \arrow[dr, bend left, "m_3"] & & & & & \\
0 \arrow[r] & K \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"'] & X_n \arrow[r] & \cdots \arrow[r] & X_3 \arrow[rr, "f_3"'] \arrow[ur, bend left, "e_3"] & & X_2 \arrow[rr, "f_2"] \arrow[dr, bend right, "e_2"'] & & X_1 \arrow[r, "f_1"'] & Q \arrow[r] & 0 \\
 & & & & & & & & I_2 \arrow[ur, bend right, "m_2"'] & & &
\end{tikzcd}
$$

we use, we get, by universal property they are the same.

Result of the syzygy

Theorem (Syzygy Theorem)

In a category with kernels and cokernels, with pullback-stable normal epimorphisms and with enough normal-projectives, for each $n \geq 1$ and any objects Q and K, there are surjections

$$\mathrm{Ext}^1(\Omega^n(Q), K) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{Ext}^n(\Omega(Q), K) \twoheadrightarrow \mathrm{Ext}^{n+1}(Q, K).$$

In other words, Ext^n is small whenever Ext^1 is small.

Varieties of algebras

In short, you have a theory of operations acting on a naive set. C is a variety of algebra if C is composed of all models of the theory and morphisms preserving the theory.

Examples

- Groups
- Lie algebra over a ring
- The category of loops
- Monoids

Why varieties of algebras?

[Bourn and Janelidze, 2003]

Theorem

Let $\mathcal{V}$ be a variety of algebras. $\mathcal{V}$ is semi-abelian if and only if the free algebra over the empty set is a singleton (in other words, there is a unique nullary operation, a constant denoted 0), and there exist

- *an integer $\ell \geq 1$,*
- *ℓ binary operations α_i such that $\alpha_i(x, x) = 0$ for all $i = 1, \ldots, \ell$, and*
- *an $(\ell + 1)$-ary operation β such that $\beta(\alpha_1(x, y), \ldots, \alpha_\ell(x, y), y) = x$.*

Why varieties of algebras

Corollary

Let $\mathcal{V}$ be a semi-abelian variety of algebras and ℓ an integer as in Theorem 2. Then any short exact sequence

$$0 \longrightarrow K \xrightarrow{k} X \xrightarrow{q} Q \longrightarrow 0$$

is a retract over Q of the short exact sequence

$$0 \longrightarrow K^{\ell} \xrightarrow{(1_{K^{\ell}},0)} K^{\ell} \times Q \xrightarrow{\pi_Q} Q \longrightarrow 0$$

in the category of pointed sets.

This, borrowed from [Peschke and Van der Linden, 2024], is a strengthening of Proposition 3.3 in [Clementino et al., 2015].

What it means for us

From this we deduce:

Theorem

For any two algebras K, Q in a semi-abelian variety $\mathcal{V}$, the conglomerate $\mathrm{Ext}^1(Q, K)$ is small.
Furthermore every variety of algebra has enough normal projective, so Ext^n is small as well.

Idea of proof: semi abelian imply any 1-extension is equivalent to a retract of a set, with a choice of structure on top. We propagate the smallness from there.

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The End

Thanks for listening.

Full article at:

References

Bourn, D. and Janelidze, G. (2003).
Characterization of protomodular varieties of universal algebras.
Theory Appl. Categ., 11(6):143–147.

Clementino, M. M., Montoli, A., and Sousa, L. (2015).
Semidirect products of (topological) semi-abelian algebras.
J. Pure Appl. Algebra, 219:183–197.

Mattenet, S., der Linden, T. V., and Jungers, R. M. (2025).
The cohomology objects of a semi-abelian variety are small.

Peschke, G. and Van der Linden, T. (2024).
A homological view of categorical algebra.
Preprint `arXiv:2404.15896`.

Bonus slide: 2-Extension

Some talk about double extensions

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & X_2' \arrow[r] \arrow[d] & I_2' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_2 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Theorem

In a semi-abelian variety $\mathcal{V}$, the conglomerate $2\text{-}\mathrm{Ext}(Q, K)$ of all double extensions between any two objects K and Q is small.

Bonus slide: Schreier monoid

The category of monoid is NOT semi abelian. Nevertheless one can still show

Theorem

For any monoids K, Q, the conglomerate $\mathrm{Ext}^1_{\underline{S}}(Q, K)$ of **Schreier** *extensions is small.*

Note that a step is missing in the proof for Ext^n as the projective object has no reason to admit a Schreier exact sequence.