Fibrational approach to Grandis exactness for 2-categories

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joint work with Elena Caviglia and Zurab Janelidze

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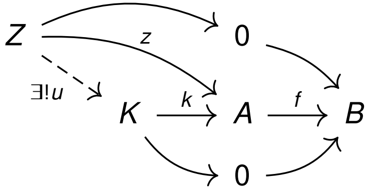
E. Caviglia, Z. Janelidze, and L. Mesiti, Fibrational approach to Grandis exactness for 2-categories, arXiv:2504.01011, 2025.

Kernels and cokernels

Let $\mathbb{C}$ be a pointed category, i.e. a category with a zero object 0.

Definition.

A **kernel** of a morphism $f\colon A \to B$ in $\mathbb{C}$ is a morphism $k\colon K \to A$ in $\mathbb{C}$ which is universal with the property that **$f \circ k$ factors through 0**.



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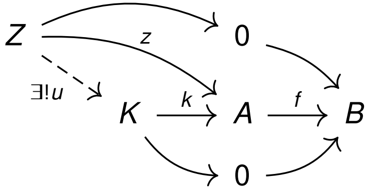
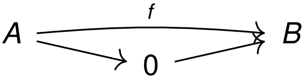


Equivalently, the kernel of f is given by the **equalizer** of

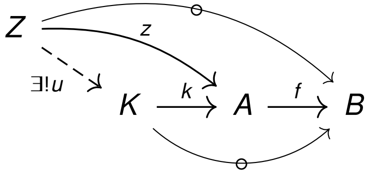


Kernels and cokernels

Let $\mathbb{C}$ be a category equipped with a (bisided) **ideal** $\mathcal{N}$ of morphisms, called **null morphisms**. Ideal means that $a \circ g \circ b \in \mathcal{N}$ for all $g \in \mathcal{N}$.

Definition.

An **$\mathcal{N}$-kernel** of a morphism $f\colon A \to B$ in $\mathbb{C}$ is a morphism $k\colon K \to A$ in $\mathbb{C}$ which is universal with the property that **$f \circ k$ is a null morphism**.



Short exact sequences

Let $\mathbb{C}$ be a category equipped with an ideal $\mathcal{N}$ of null morphisms.

Definition.

A **short exact sequence** is a sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that $f = \mathsf{Ker}(g)$ and $g = \mathsf{Coker}(f)$. In particular, $g \circ f$ is null.

Abelian categories offer a nice categorical framework which captures short exact sequences and homological algebra in general.

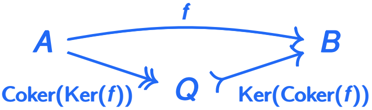
Notions of exact categories try to capture as much homological algebra as possible without requiring an additive category.

Definition.

A **Grandis exact category** is a category $\mathbb{C}$ equipped with an ideal $\mathcal{N}$ of null morphisms such that

- $\mathcal{N}$ is a closed ideal;
- $\mathbb{C}$ has all $\mathcal{N}$-kernels and $\mathcal{N}$-cokernels;
- every morphism f factorizes as an $\mathcal{N}$-cokernel followed by an $\mathcal{N}$-kernel.

It follows that



Puppe exact coincides with pointed Grandis exact w.r.t. the zero ideal.

A fibrational approach to Grandis exactness

Let $\mathbb{C}$ be a category and let $(\mathcal{E}, \mathcal{M})$ be an (orthogonal) **factorization system** on $\mathbb{C}$. Consider $\mathfrak{E}$ and $\mathfrak{M}$ the full subcategories of the arrow category of $\mathbb{C}$ on morphisms in $\mathcal{E}$ and $\mathcal{M}$ respectively.

Proposition.

The codomain functor $\mathrm{cod}\colon \mathfrak{M} \to \mathbb{C}$ *is an opfibration, called the* ***opfibration of subobjects*** *relative to* $(\mathcal{E}, \mathcal{M})$.
Dually, the domain functor $\mathrm{dom}\colon \mathfrak{E} \to \mathbb{C}$ *is a fibration, called the* ***fibration of quotients*** *relative to* $(\mathcal{E}, \mathcal{M})$.

$$\begin{array}{ccc} A & \overset{\ell^{f\circ m}}{\twoheadrightarrow} & Q \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle r^{f\circ m}} \\ B & \underset{f}{\longrightarrow} & C \end{array} \qquad\qquad \begin{array}{c} \mathfrak{M} \\ \downarrow{\scriptstyle \mathrm{cod}} \\ \mathbb{C} \end{array}$$

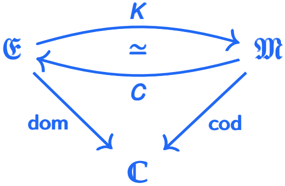
A fibrational approach to Grandis exactness

Theorem ([JanWei16]).

Let $\mathbb{C}$ be a category. The following conditions are equivalent:

(*i*) *$\mathbb{C}$ has the structure of a Grandis exact category;*

(*ii*) *There exists a proper factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathbb{C}$ and an equivalence over $\mathbb{C}$*



between the fibration of quotients and the opfibration of subobjects relative to $(\mathcal{E}, \mathcal{M})$.

[JanWei16] Z. Janelidze and T. Weighill, Duality in non-abelian algebra II, Journal of Homotopy and Related Structures 11, 553–570, 2016.

Towards dimension 2

An ideal of null morphisms in a category $\mathbb{C}$ is equivalently a **subprofunctor** N of the profunctor $\mathrm{Hom}(-,-)$, i.e.

$$\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \underset{\mathrm{Hom}(-,-)}{\overset{N}{\rightrightarrows}} \mathbf{Set} \quad (\Downarrow \nu) \qquad \text{with injective components.}$$

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Definition.

A **2-ideal** $\mathcal{N}$ of null morphisms and null 2-cells in a 2-category $\mathfrak{L}$ is a pair (N, ν) where $N \colon \mathfrak{L}^{\mathrm{op}} \times \mathfrak{L} \to \mathbf{Cat}$ is a normal pseudofunctor and ν is

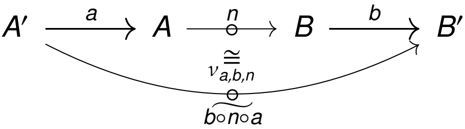
$$\mathfrak{L}^{\mathrm{op}} \times \mathfrak{L} \; \underset{\mathrm{Hom}(-,-)}{\overset{N}{\Downarrow \nu}} \; \mathbf{Cat} \qquad \text{injective on objects and faithful pseudo natural transformation}$$

2-ideals of null morphisms and null 2-cells

Proposition (C.J.M.).

Let $\mathfrak{L}$ be a 2-category. A 2-ideal $\mathcal{N}$ in $\mathfrak{L}$ can be equivalently given as

- *a class of null morphisms and a class of null 2-cells between them, forming subcategories of the hom-categories;*
- *isomorphic 2-cells*



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$$A' \xrightarrow{\;a\;} A \xrightarrow{\;n\;} B \xrightarrow{\;b\;} B', \qquad \nu_{a,b,n}\colon b\circ n\circ a \overset{\cong}{\Longrightarrow} \widetilde{b\circ n\circ a}$$

such that

(1) $\nu_{\mathrm{id},\mathrm{id},n} = \mathrm{id}$ *and the ν's respect composition up to an iso null 2-cell;*

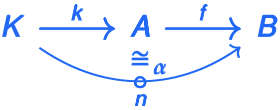
(2) *"$b * \mu * a$" is a null 2-cell for every null 2-cell μ and morphisms a, b;*

(3) *"$\beta * n * \alpha$" is a null 2-cell for every null morphism n and 2-cells α, β.*

2-kernels and 2-cokernels

Definition.

An ***$\mathcal{N}$*-2-kernel** of $f\colon A \to B$ in $\mathfrak{L}$ is a morphism $K \xrightarrow{k} A$ together with



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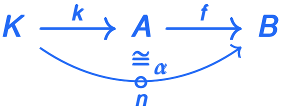

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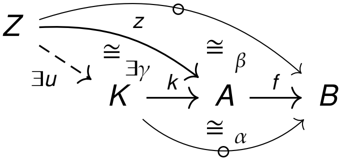
(1) for every $Z \xrightarrow{z} A$ s.t. $f \circ z$ iso to null via β, there exist $Z \xrightarrow{u} K$ and an iso 2-cell γ



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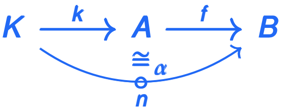

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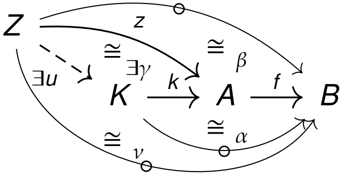
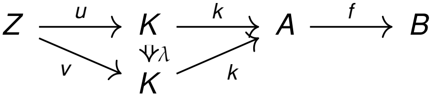


2-kernels and 2-cokernels

(2) for every $u, v\colon Z \to K$ and every 2-cell $\lambda\colon k \circ u \Rightarrow k \circ v$ such that



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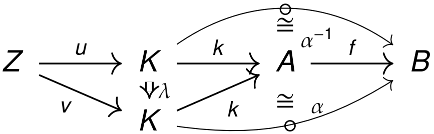

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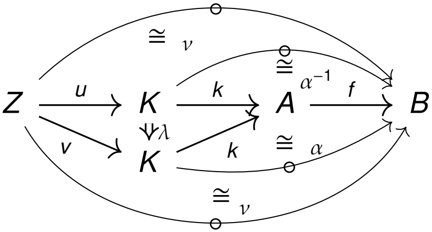

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The 2-pointed case

Example.

Let $\mathfrak{L}$ be a **2-pointed** 2-category, in the sense that there exists a (chosen) object 0 s.t. $\mathfrak{L}(A, 0) \simeq 1 \simeq \mathfrak{L}(0, A)$.

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$$\begin{array}{ccccc} & \overset{t}{\nearrow} & 0 & \overset{i}{\searrow} & \\ A & \cong & \| & \cong & B. \\ & \underset{t'}{\searrow} & 0 & \underset{i'}{\nearrow} & \end{array} \qquad (1)$$

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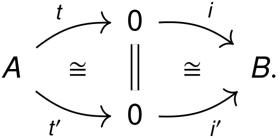
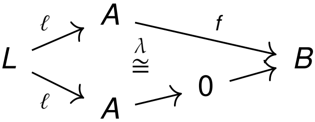


(1)

The 2-kernel of $f\colon A \to B$ is given by the **biisoinserter** of $f\colon A \to B$ and a chosen null morphism $A \to 0 \to B$.



Closed 2-ideals

Proposition.

Let $\mathbb{C}$ be a category and $\mathcal{N}$ be an ideal of null morphisms in $\mathbb{C}$. Assume that $\mathbb{C}$ has all kernels and cokernels. TFAE:

(*i*) *$\mathcal{N}$ is closed, i.e. every null morphism factors through a null object;*

(*ii*) *All kernels k reflect null morphisms, i.e. if $k \circ f$ is null then f is null;*

(*iii*) *All cokernels c coreflect null morphisms, i.e. if $f \circ c$ is null then f is null.*

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Definition.

Let $\mathfrak{L}$ be a 2-category and $\mathcal{N} = (N, \nu)$ be a 2-ideal of null morphisms and null 2-cells in $\mathfrak{L}$ s.t. $\mathfrak{L}$ has all $\mathcal{N}$-2-kernels and $\mathcal{N}$-2-cokernels. We call $\mathcal{N}$ **closed** if all $\mathcal{N}$-2-kernels reflect null morphisms and null 2-cells and all $\mathcal{N}$-2-cokernels coreflect null morphisms and null 2-cells.

Weak 2-opfibration of $\mathcal{M}$-subobjects

Definition.

Let $P\colon \mathfrak{K} \to \mathfrak{L}$ be a 2-functor. We call P a **weak 2-fibration** if

- for every $E \in \mathfrak{K}$ and every $f\colon L \to P(E)$ in $\mathfrak{L}$, there exists a **2-cartesian lifting** of f to E, in a bicategorical sense;
- P is **locally an isofibration**, i.e. for every $E, E' \in \mathfrak{K}$ the induced functor $\mathfrak{K}(E, E') \to \mathfrak{L}(P(E), P(E'))$ is an isofibration.

Proposition (C.J.M.).

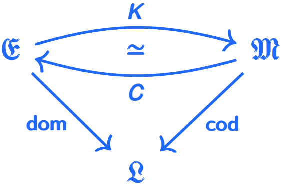
Let $(\mathcal{E}, \mathcal{M})$ be a factorization system on a 2-category $\mathfrak{L}$. Consider $\mathfrak{E}$ and $\mathfrak{M}$ the full sub-2-categories of the ***pseudo arrow category*** *of $\mathfrak{L}$ on morphisms in $\mathcal{E}$ and $\mathcal{M}$ respectively. Then $\operatorname{cod}\colon \mathfrak{M} \to \mathfrak{L}$ is a weak 2-opfibration and $\operatorname{dom}\colon \mathfrak{E} \to \mathfrak{L}$ is a weak 2-fibration.*

Main theorem

Theorem (C.J.M.).

Let $\mathfrak{L}$ be a 2-category. The following conditions are equivalent:

(*i*) *There exists a* $(1,1)$*-proper factorization system* $(\mathcal{E}, \mathcal{M})$ *on the 2-category* $\mathfrak{L}$ *and a* ***biequivalence*** *over* $\mathfrak{L}$

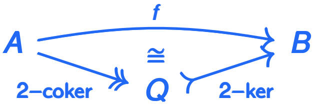


between the weak 2-fibration of 2-quotients relative to $(\mathcal{E}, \mathcal{M})$ *and the weak 2-opfibration of 2-subobjects relative to* $(\mathcal{E}, \mathcal{M})$*, with* K, C *normal pseudofunctors.*

Main theorem

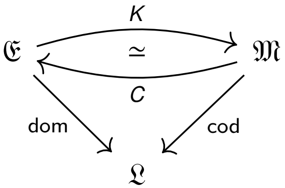
(ii) $\mathfrak{L}$ has a 2-ideal $\mathcal{N}$ of null morphisms and null 2-cells such that

- $\mathfrak{L}$ has all $\mathcal{N}$-2-kernels and $\mathcal{N}$-2-cokernels;
- $\mathcal{N}$ is a closed 2-ideal;
- every $\mathcal{N}$-2-kernel m is an $\mathcal{N}$-2-kernel of its $\mathcal{N}$-2-cokernel c_m, with structure isomorphic 2-cell given up to an iso null 2-cell by the $\mathcal{N}$-2-cokernel c_m of m; and dually for $\mathcal{N}$-2-cokernels;
- every morphism f in $\mathfrak{L}$ factorizes up to isomorphism as an $\mathcal{N}$-2-cokernel followed by an $\mathcal{N}$-2-kernel



Idea of the proof

$(ii) \Rightarrow (i)$. Define $\mathcal{E} = \{\mathcal{N}\text{-2-cokernels}\}$ and $\mathcal{M} = \{\mathcal{N}\text{-2-kernels}\}$.
This gives a $(1,1)$-proper factorization system on the 2-category $\mathfrak{L}$.
We then define ***K*** **and** ***C*** **to calculate** $\mathcal{N}$**-2-kernels and** $\mathcal{N}$**-2-cokernels** respectively.



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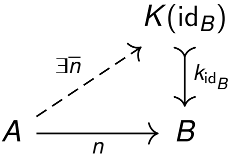

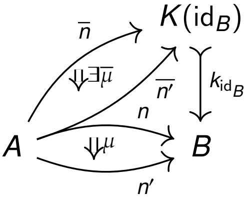

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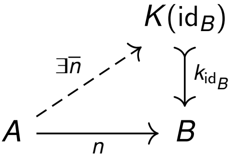
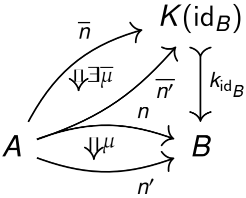
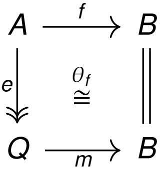
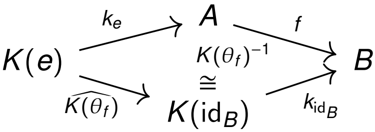




The kernel of $f\colon A \to B$ is given by applying K to its epi part:





2-dimensional Grandis exactness

Definition.

We call a 2-category $\mathfrak{L}$ **Grandis 2-exact** if it satisfies one of the two equivalent conditions of the main theorem.

Definition.

We call a 2-pointed 2-category $\mathfrak{L}$ **Puppe 2-exact** if it satisfies one of the following two equivalent conditions:

(i) condition (*i*) of the main theorem with K and C sending identities to morphisms from 0 and into 0 respectively;

(ii) condition (*ii*) of the main theorem with $\mathcal{N}$ given by the 2-ideal (0) canonically associated to $\mathfrak{L}$.

Interesting facts

There are a few differences from what happens in dimension 1:

- Not all 2-monos in a 2-Puppe exact 2-category seem to be 2-kernels;
- Having a closed 2-ideal is stronger than having that all null morphisms factor through a null object;
- The zero 2-ideal of a 2-pointed 2-category does not seem to be automatically closed;
- Not every 2-kernel is the 2-kernel of its 2-cokernel.

The motivating example

By the results of [JanRei25], we can conclude the following:

Theorem.

*The **2-category of abelian categories** with suitably chosen functors as morphisms, called **Serre functors**, is Grandis 2-exact (actually, Puppe 2-exact).*

[JanRei25] Z. Janelidze and Ü. Reimaa, Serre functors between Puppe exact categories and 2-dimensional exactness, in preparation, 2025.