Chase–Sweedler Galois theory in additive monoidal categories

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[HAaGT] Chase, S. and Sweedler, M. *Hopf Algebras and Galois Theory*. Lecture Notes in Mathematics. Springer. (1969)

Concrete algebraic

Abstract categorical

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Concrete algebraic

(Co)monoids in an additive symmetric monoidal category

Abstract categorical

 \mathbf{Alg}_R

 \mathbf{Coalg}_R

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 $\mathbf{Coalg}_R = \mathrm{CocComon}(\mathbf{Mod}_R)$

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- \otimes_R additive in each argument

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Duality

 $\mathbf{Alg}_R^{\mathrm{fgp}}$ is dual to $\mathbf{Coalg}_R^{\mathrm{fgp}}$ [HAaGT]

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[HAaGT]

fgp: underlying module is finitely-generated projective

 $\operatorname{CocComon}(\mathbb{C}^{\operatorname{fgp}})$ is dual to $\operatorname{CMon}(\mathbb{C}^{\operatorname{fgp}})$

We call an object C in $\mathbb C$ finitely-generated projective if there exists a split epimorphism $\varphi: \bigoplus_{i=1}^n I \to C$

$$[-, I] : \operatorname{CocComon}(\mathbb{C}^{\operatorname{fgp}})^{\operatorname{op}} \to \operatorname{CMon}(\mathbb{C}^{\operatorname{fgp}})$$

Hopf algebras

A group object in \mathbf{Coalg}_R is a cocommutative Hopf algebra.

Hopf algebras

A group object in $Coalg_R$ is a cocommutative Hopf algebra.

For a finite group G, R[G] is the group algebra with coefficients in R, namely:

$$R[G] = \sum_{g \in G} r_g g$$

R[G] is a group object in \mathbf{Coalg}_R

[HAaGT]

Abstract Hopf algebras & abstract group algebras

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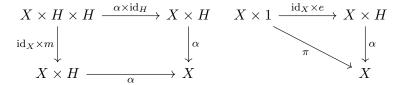
$$I[-]: \mathbf{Set} \to \operatorname{CocComon}(\mathbb{C})$$
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- I[-] preserves products, sends group objects to group objects
- A group object in Set is an algebraic group
- For a finite group G, I[G] is a group in $CocComon(\mathbb{C})$

H-objects

Let $\mathbb A$ be a category with finite products (terminal object 1) and (H,m,e,λ) a group in $\mathbb A.$

A H-object in $\mathbb A$ is an object X with $\alpha: X \times H \to X$ s.t.



Galois H-objects

An object X in $\mathbb A$ is called **faithful** if for all $f:A\to B$ in $\mathbb A$, $\operatorname{id}_X\times f \text{ is an iso.} \implies f \text{ is an iso.}$

A H-object X is a **Galois** H-object if X is faithful and

$$\gamma = \langle \pi_X, \alpha \rangle : X \times G \to X \times X$$

is an isomorphism.

Galois extensions

- R[G] is a group in \mathbf{Coalg}_R
- The dual $R[G]^*$ is a cogroup in \mathbf{Alg}_R

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- I[G] is a group in $\operatorname{CocComon}(\mathbb{C})$
- $I[G]^* \coloneqq [I[G], I]$ is a cogroup in $\mathrm{CMon}(\mathbb{C})$

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