Chase–Sweedler Galois theory in additive monoidal categories

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[HAaGT] Chase, S. and Sweedler, M. *Hopf Algebras and Galois Theory*. Lecture Notes in Mathematics. Springer. (1969)

Concrete algebraic

Abstract categorical

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Concrete algebraic

(Co)monoids in an additive symmetric monoidal category

Abstract categorical

Intermediate world

$\mathbf{Alg}_R$

$\mathbf{Coalg}_R$

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[HAaGT]

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fgp: underlying module is finitely-generated projective

$\mathrm{CocComon}(\mathbb{C}^{\mathrm{fgp}})$ is dual to $\mathrm{CMon}(\mathbb{C}^{\mathrm{fgp}})$

We call an object C in $\mathbb{C}$ **finitely-generated projective** if there exists a split epimorphism $\varphi : \bigoplus_{i=1}^{n} I \to C$

$$[-, I] : \mathrm{CocComon}(\mathbb{C}^{\mathrm{fgp}})^{\mathsf{op}} \to \mathrm{CMon}(\mathbb{C}^{\mathrm{fgp}})$$

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For a finite group G, $R[G]$ is the group algebra with coefficients in R, namely:

$$R[G] = \sum_{g \in G} r_g g$$

$R[G]$ is a group object in $\mathbf{Coalg}_R$ **[HAaGT]**

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- $I[-]$ preserves products, sends group objects to group objects
- A group object in $\mathbf{Set}$ is an algebraic group
- For a finite group G, $I[G]$ is a group in $\mathrm{CocComon}(\mathbb{C})$

H-objects

Let $\mathbb{A}$ be a category with finite products (terminal object 1) and (H, m, e, λ) a group in $\mathbb{A}$.

A ***H*-object** in $\mathbb{A}$ is an object X with $\alpha : X \times H \to X$ s.t.

$$\begin{array}{ccc} X \times H \times H & \xrightarrow{\alpha \times \mathrm{id}_H} & X \times H \\ {\scriptstyle \mathrm{id}_X \times m} \downarrow & & \downarrow {\scriptstyle \alpha} \\ X \times H & \xrightarrow[\alpha]{} & X \end{array} \qquad \begin{array}{ccc} X \times 1 & \xrightarrow{\mathrm{id}_X \times e} & X \times H \\ & {\scriptstyle \pi} \searrow & \downarrow {\scriptstyle \alpha} \\ & & X \end{array}$$

Galois H-objects

An object X in $\mathbb{A}$ is called **faithful** if for all $f : A \to B$ in $\mathbb{A}$,

$$\mathrm{id}_X \times f \text{ is an iso.} \implies f \text{ is an iso.}$$

A H-object X is a **Galois H-object** if X is faithful and

$$\gamma = \langle \pi_X, \alpha \rangle : X \times G \to X \times X$$

is an isomorphism.

Galois extensions

- $R[G]$ is a group in $\mathbf{Coalg}_R$
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[HAaGT]

- $I[G]$ is a group in $\mathrm{CocComon}(\mathbb{C})$
- $I[G]^* \coloneqq [I[G], I]$ is a cogroup in $\mathrm{CMon}(\mathbb{C})$

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