Homological lemmas in a non-pointed context

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Joint work with Andrea Cappelletti



Definition (Bourn)

A finitely complete category $\mathcal C$ is protomodular if the change-of-base functors of the codomain functor $\mathsf{Pt}(\mathcal C) \to \mathcal C$ are conservative.

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A pointed category $\mathcal C$ is protomodular if and only if the split short five lemma holds.

Adding regularity, the short five lemma holds. This led to the notion of homological category (Borceux-Bourn).

Theorem (Bourn)

Given a diagram with exact columns in an homological category \mathcal{C} :

$$K \xrightarrow{u} K' \xrightarrow{u'} K''$$

$$k \downarrow \qquad \qquad \downarrow k' \qquad \qquad \downarrow k''$$

$$A \xrightarrow{a} A' \xrightarrow{a'} A''$$

$$f \downarrow \qquad \qquad \downarrow f' \qquad \qquad \downarrow f''$$

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Upper: rows 2 and 3 exact \Rightarrow row 1 exact.

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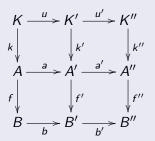
Given a diagram with exact columns in an homological category C:

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Upper: rows 2 and 3 exact \Rightarrow row 1 exact. Lower: rows 1 and 2 exact \Rightarrow row 3 exact.

Theorem (Bourn)

Given a diagram with exact columns in an homological category C:



Upper: rows 2 and 3 exact \Rightarrow row 1 exact.

Lower: rows 1 and 2 exact \Rightarrow row 3 exact.

Middle: rows 1 and 3 exact and $a'a = 0 \Rightarrow row 2$ exact.

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The denormalized upper and lower nine lemmas are equivalent for regular categories, and they characterize Goursat categories (Gran-Rodelo).

The denormalized middle nine lemma always holds.

Unified treatment (Gran-Janelidze-Rodelo): $\mathcal C$ lex, $\mathcal N$ ideal of morphisms.

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Definition (Gran-Janelidze-Rodelo)

 $\mathcal C$ is star-regular if every regular epi is the coequalizer of a star. $\mathcal C$ has enough trivial objects if every $f \in \mathcal N$ factors through a trivial object X ($1_X \in \mathcal N$), and X trivial $\Rightarrow X \times X$ trivial.

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If C is star-regular with enough trivial objects, then: Upper nine lemma \Leftrightarrow Lower nine lemma. Middle nine lemma \Leftrightarrow Short five lemma (under mild assumptions).

This includes the classical case in (quasi) pointed categories and the denormalized case but it fails to describe the situation of algebraic structures with more than one constant, like unitary rings.

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The definition of \mathcal{Z} -cokernel is dual.



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the topological models of protomodular varieties with more than one constant.

In this context, $\mathcal{Z}\text{-kernels}$ always exist, and they are obtained as pullbacks

$$\begin{array}{ccc}
K[f] & \longrightarrow Z(B) \\
\downarrow & & \downarrow \\
A & \longrightarrow B.
\end{array}$$

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Moreover, in a pointed category, given a pullback

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
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one has $Ker(f) \cong Ker(g)$ (and the converse holds if C is protomodular and f is a regular epi). In our context this is false.

Short \mathcal{Z} -exact sequences



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In Ring
$$\mathbb{Z} \longrightarrow \mathbb{Z}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

However, it becomes true if, in the square $A \xrightarrow{f} B$ $\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad$

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Short \mathcal{Z} -exact sequences

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Definition

A short Z-exact sequence is a diagram

$$K \xrightarrow{k} A \xrightarrow{f} B$$

such that f is a regular epi and k is a \mathbb{Z} -kernel of f.

The short five lemma

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Proposition

Let $\mathcal C$ be a regular protomodular category with $0 \to 1$ regular epi. In a commutative diagram

in which the rows are short \mathcal{Z} -exact sequences, if u and b are isomorphisms, then a is an isomorphism, too.

Proposition

Given a diagram with Z-exact columns:

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Upper: if $Z(A'') \cong Z(B'') \cong Z(B')$, rows 2 and 3 Z-exact \Rightarrow row 1 Z-exact.

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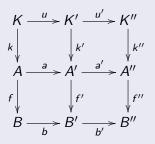
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Upper: if $Z(A'') \cong Z(B'') \cong Z(B')$, rows 2 and 3 \mathbb{Z} -exact \Rightarrow row 1 \mathbb{Z} -exact.

Lower: if $Z(A'') \cong Z(B'') \cong Z(B') \cong Z(B)$, rows 1 and 2 Z-exact \Rightarrow row 3 Z-exact.

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Given a diagram with Z-exact columns:



Upper: if $Z(A'') \cong Z(B'') \cong Z(B')$, rows 2 and 3 \mathbb{Z} -exact \Rightarrow row 1 \mathbb{Z} -exact.

Lower: if $Z(A'') \cong Z(B'') \cong Z(B') \cong Z(B)$, rows 1 and 2 \mathbb{Z} -exact \Rightarrow row 3 \mathbb{Z} -exact.

Middle: if $Z(A'') \cong Z(B'') \cong Z(B')$, rows 1 and 3 Z-exact and $a'a \in \mathcal{N}_Z \Rightarrow row$ 2 Z-exact.