Homological lemmas in a non-pointed context

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Homological categories

Definition (Bourn)

A finitely complete category $\mathcal{C}$ *is protomodular if the change-of-base functors of the codomain functor* $\mathrm{Pt}(\mathcal{C}) \to \mathcal{C}$ *are conservative.*

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A pointed category $\mathcal{C}$ is protomodular if and only if the split short five lemma holds.

Adding regularity, the short five lemma holds. This led to the notion of homological category (Borceux-Bourn).

The nine lemma

Theorem (Bourn)

Given a diagram with exact columns in an homological category $\mathcal{C}$:

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\begin{array}{ccccc}
K & \xrightarrow{u} & K' & \xrightarrow{u'} & K'' \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle k''} \\
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Upper: rows 2 and 3 exact $\Rightarrow$ *row 1 exact.*

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Upper: rows 2 and 3 exact $\Rightarrow$ row 1 exact.
Lower: rows 1 and 2 exact $\Rightarrow$ row 3 exact.
Middle: rows 1 and 3 exact and $a'a = 0 \Rightarrow$ row 2 exact.

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The denormalized middle nine lemma always holds.

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A short star-exact sequence is a diagram $K \underset{\kappa_2}{\overset{\kappa_1}{\rightrightarrows}} A \xrightarrow{f} B$ where (κ_1, κ_2) is a star-kernel of f and $f = \mathrm{coeq}(\kappa_1, \kappa_2)$.

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Definition (Gran-Janelidze-Rodelo)

$\mathcal{C}$ is star-regular if every regular epi is the coequalizer of a star. $\mathcal{C}$ has enough trivial objects if every $f \in \mathcal{N}$ factors through a trivial object X ($1_X \in \mathcal{N}$), and X trivial $\Rightarrow X \times X$ trivial.

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This includes the classical case in (quasi) pointed categories and the denormalized case but it fails to describe the situation of algebraic structures with more than one constant, like unitary rings.

Our context

Let $\mathcal{C}$ be a regular category with 0 such that the unique morphism $0 \to 1$ is a regular epimorphism.

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The definition of $\mathcal{Z}$-cokernel is dual.

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the topological models of protomodular varieties with more than one constant.

In this context, $\mathcal{Z}$-kernels always exist, and they are obtained as pullbacks

$$\begin{array}{ccc} K[f] & \longrightarrow & \mathrm{Z}(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B. \end{array}$$

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Moreover, in a pointed category, given a pullback

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Short $\mathcal{Z}$-exact sequences

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Definition

A short $\mathcal{Z}$-exact sequence is a diagram
$$K \xrightarrow{k} A \xrightarrow{f} B$$
such that f is a regular epi and k is a $\mathcal{Z}$-kernel of f.

The short five lemma

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Proposition

Let $\mathcal{C}$ be a regular protomodular category with $0 \to 1$ regular epi. In a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ K' & \xrightarrow[k']{} & A' & \xrightarrow[f']{} & B' \end{array}$$

in which the rows are short $\mathcal{Z}$-exact sequences, if u and b are isomorphisms, then a is an isomorphism, too.

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Middle: if $\mathrm{Z}(A'') \cong \mathrm{Z}(B'') \cong \mathrm{Z}(B')$, rows 1 and 3 $\mathcal{Z}$-exact and $a'a \in \mathcal{N}_{\mathcal{Z}}$ $\Rightarrow$ row 2 $\mathcal{Z}$-exact.