Profinite completions and clones

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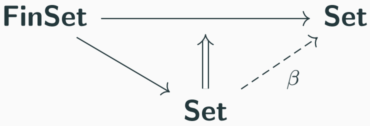
Ultrafilters and codensity

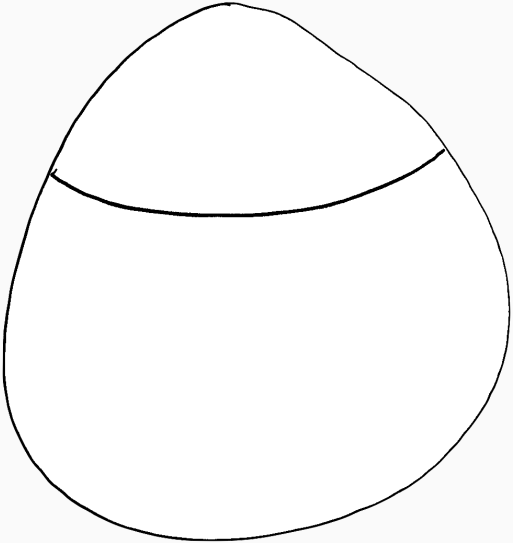
Theorem (Kennison and Gildenhuys). The ultrafilter monad β is the codensity monad of the inclusion **FinSet** $\to$ **Set**, i.e. the right Kan extension



This means that we have

$$\beta X \quad \cong \quad \int_{F \in \mathbf{FinSet}} F^{\mathbf{Set}(X,F)}$$

Ultrafilters: playing "Guess who?"



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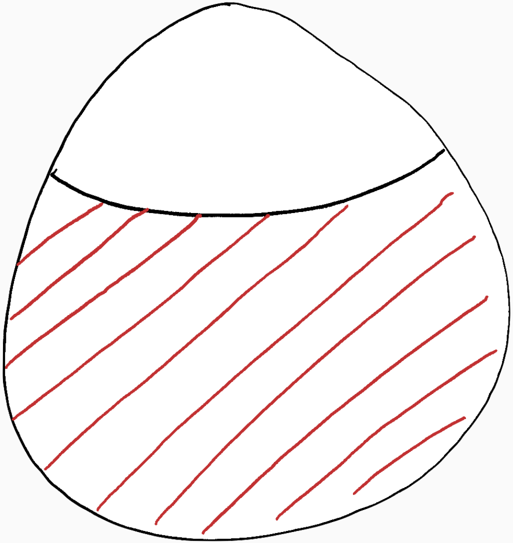

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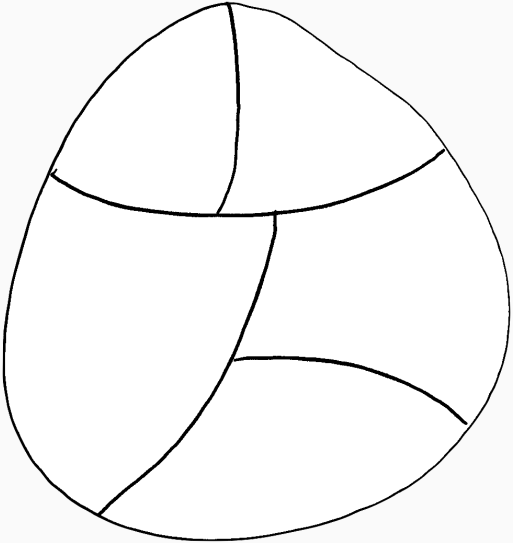

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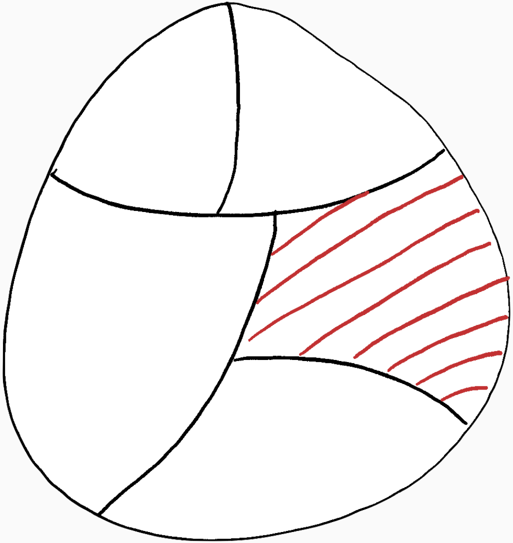

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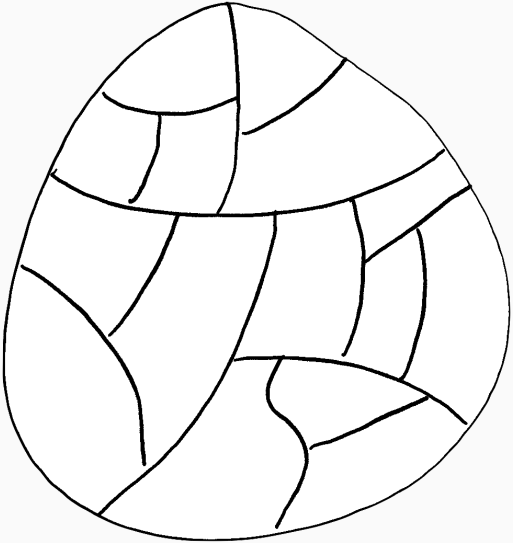

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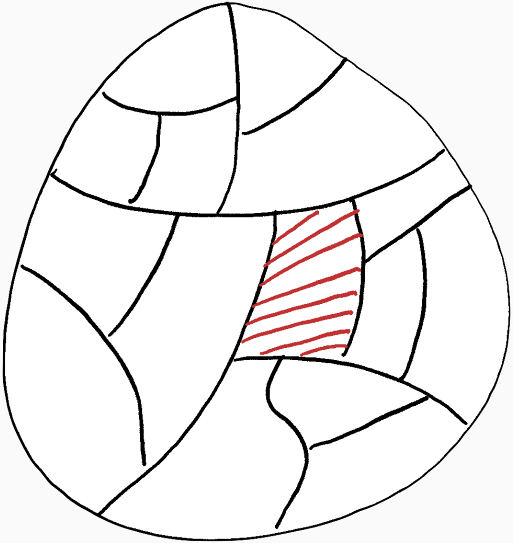

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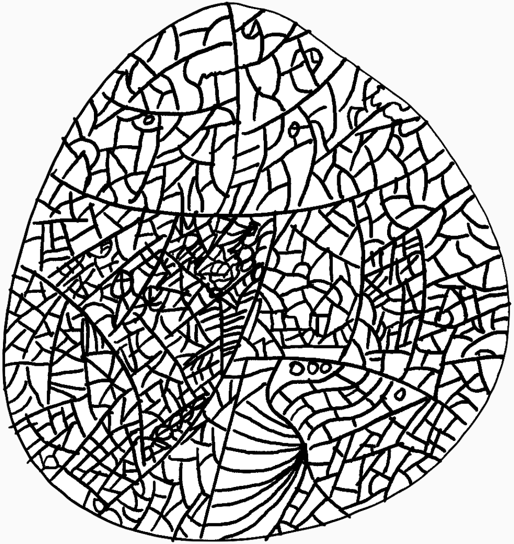

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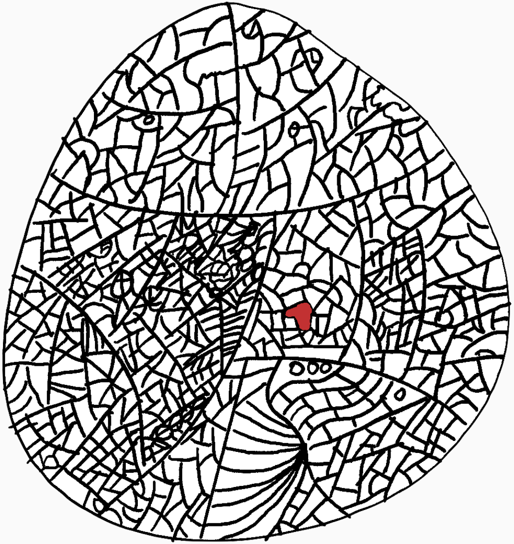
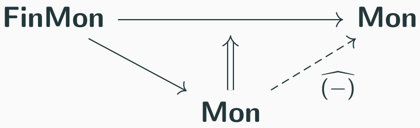


Profinite completion of monoids

We write **FinMon** for the full subcategory of **Mon** containing finite monoids.

The profinite completion of monoids is defined as the codensity monad



As before, we have

$$\widehat{M} \quad = \quad \int_{N \in \mathbf{FinMon}} N^{\mathbf{Mon}(M,N)}$$

This profinite completion is very important in automata theory.

Abstract clones

A clone C is a family of sets

$$C_n \qquad \text{where } n \text{ ranges over natural numbers}$$

together with elements representing variables

$$v_k \quad \in \quad C_n \qquad \text{for every } n, k \in \mathbb{N} \text{ such that } 1 \leq k \leq n$$

and functions representing composition

$$s_{m,n} \quad : \quad C_n \times (C_m)^n \longrightarrow C_m \qquad \text{for every } m, n \in \mathbb{N}$$

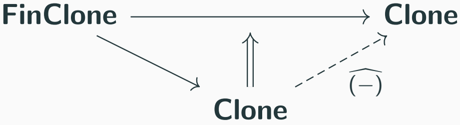
that verify some conditions. Equivalently: one-object cartesian multicategories.

Together with the appropriate morphisms, they form the category **Clone**.

Profinite completion of clones

We write **FinClone** for the full subcategory of **Clone** containing the clones D such that the sets D_n are finite for every $n \in \mathbb{N}$.

Definition. The profinite completion of clones is defined as the codensity monad



Yet again,

$$\widehat{C} \quad \cong \quad \int_{D \in \mathbf{FinClone}} D^{\mathbf{Clone}(C,D)}$$

The theory of clones

We write $T_{\textbf{Clone}}$ for the cartesian category whose objects are signatures

$$\Sigma \quad := \quad [n_1, \ldots, n_l]$$

with cartesian product being the concatenation, and whose hom-sets are

$$T_{\textbf{Clone}}(\Sigma, [m]) \quad := \quad \{\text{trees built from } \Sigma \text{ with variables among } v_1, \ldots, v_m\}$$

A special case of a theorem by Fiore shows that

$$\textbf{Mod}(T_{\textbf{Clone}}) \quad \cong \quad \textbf{Clone}$$

where $\textbf{Mod}(T)$ is the category of product-preserving functors $T \to \textbf{Set}$ for any T.

Free/forgetful adjunctions

An important observation by Lawvere is that any product-preserving functor

$$F \quad : \quad T \longrightarrow T'$$

between cartesian categories induces an adjunction

$$\mathbf{Mod}(T) \underset{(-)\circ F}{\overset{\mathrm{Lan}_F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Mod}(T')$$

Encodings of sets and monoids

The fully faithful functor $T_{\textbf{Mon}} \to T_{\textbf{Clone}}$ yields the coreflective adjunction

$$\textbf{Mon} \underset{C_1 \;\leftarrow\!\shortmid\; C}{\overset{M \;\mapsto\; (M \times \{v_1, \ldots, v_n\})_n}{\rightleftarrows}} \textbf{Clone} \qquad (\bot)$$

The fully faithful functor $T_{\textbf{Set}} \to T_{\textbf{Clone}}$ yields the coreflective adjunction

$$\textbf{Set} \underset{C_0 \;\leftarrow\!\shortmid\; C}{\overset{X \;\mapsto\; (X \sqcup \{v_1, \ldots, v_n\})_n}{\rightleftarrows}} \textbf{Clone} \qquad (\bot)$$

Parametric right adjoints

As in Soichiro's talk, we consider parametric right adjoints, i.e. functors

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{D}$$

where $\mathbf{C}$ has a terminal object 1, such that there exists a left adjoint

$$\mathbf{C} \cong \mathbf{C}/1 \underset{\perp}{\overset{\dashleftarrow}{\longrightarrow}} \mathbf{D}/F1 \longrightarrow \mathbf{D}$$

where the functor $\mathbf{C} \to \mathbf{D}/F1$ sends an object X on the morphism $F(!_X) : FX \to F1$.

Goal and illustrating example

If F is a parametric right adjoint, then it preserves all connected limits.

To relate the different profinite completions, we want our two fully faithful functors

$$\mathrm{Cl}_1 \quad : \quad \mathbf{Mon} \longrightarrow \mathbf{Clone} \qquad \text{and} \qquad \mathrm{Cl}_0 \quad : \quad \mathbf{Set} \longrightarrow \mathbf{Clone}$$

to be parametric right adjoints.

Today's talk: we focus on a simpler functor

$$\mathbf{Sgp} \longrightarrow \mathbf{Mon}$$

from semigroups to monoids, that showcases the techniques used for clones.

From semigroups to monoids

We write **Sgp** for the category of semigroups, which is equivalent to $\mathbf{Mod}(T_{\mathbf{Sgp}})$ for

$$T_{\mathbf{Sgp}}(n, 1) \quad = \quad \{\text{non-empty finite words over } \{a_1, \ldots, a_n\}\}$$

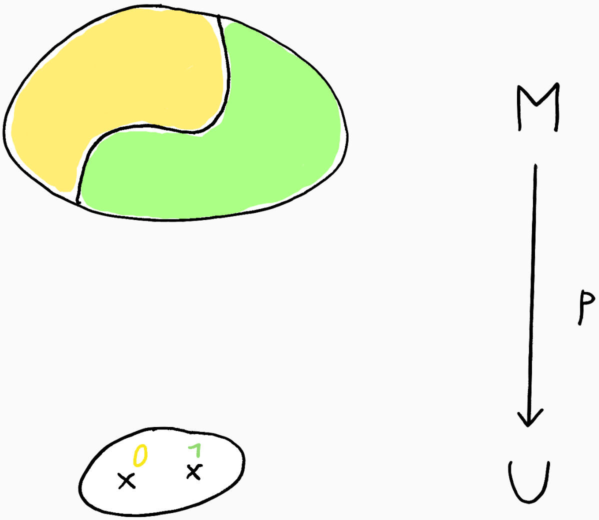
We get a faithful functor $T_{\mathbf{Sgp}} \to T_{\mathbf{Mon}}$, which yields the adjunction

$$\mathbf{Sgp} \underset{\text{forget}}{\overset{S \;\mapsto\; S \sqcup \{1\}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Mon}$$

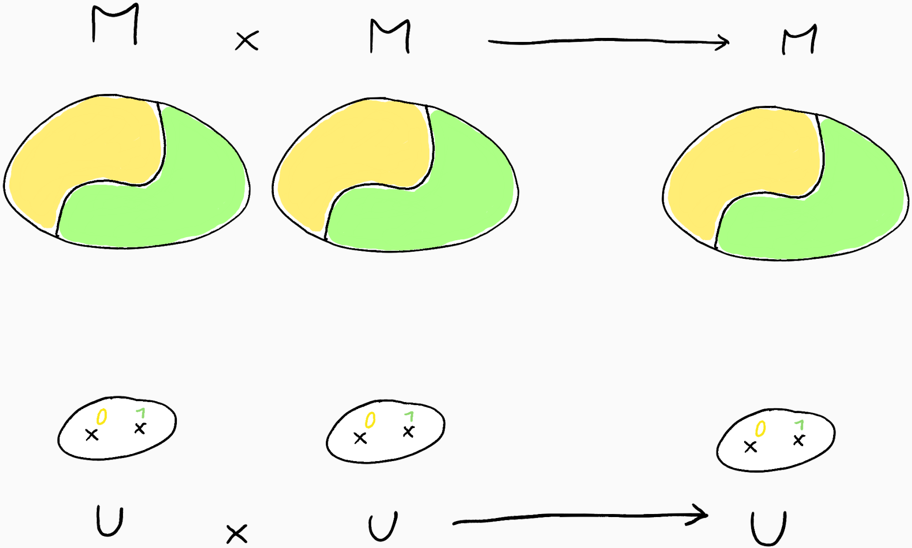
The left adjoint sends the terminal semigroup $\{0\}$ on the monoid

$$U \quad := \quad (\{0, 1\}, \times)$$

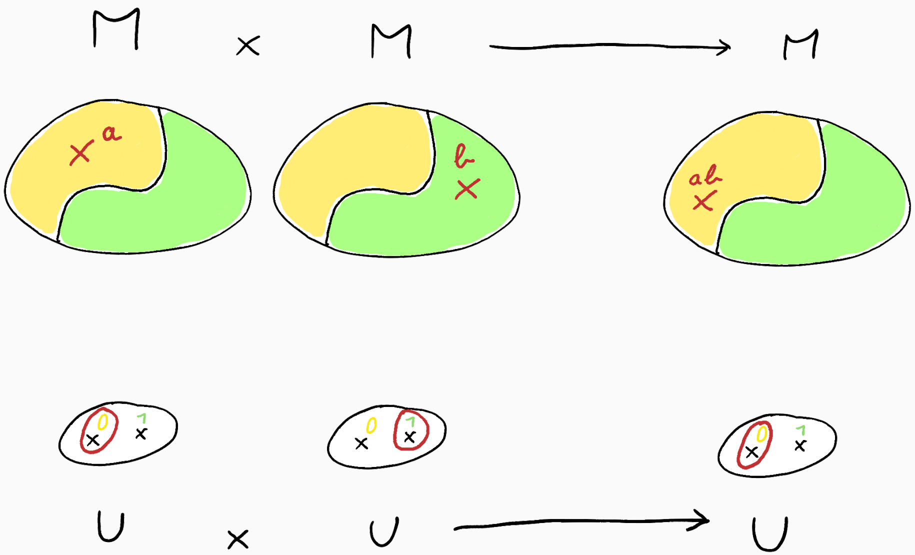
Monoids over U



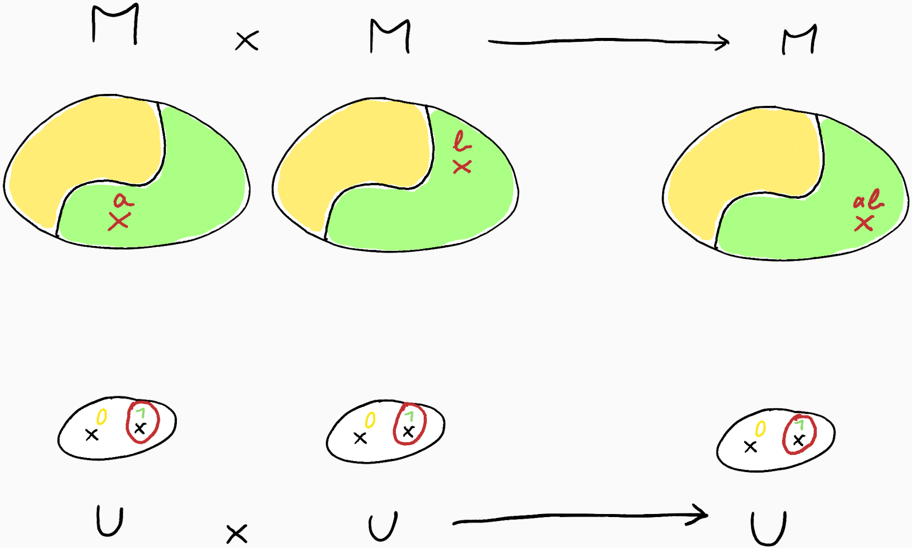
Monoids over U



Monoids over U



Monoids over U



Prime ideals in an indexed way

The data of a homomorphism

$$p \quad : \quad M \longrightarrow U := \{0, 1\}$$

can be equivalently described as two sets

$$M_0 \quad := \quad p^{-1}\{0\} \qquad \text{and} \qquad M_1 \quad := \quad p^{-1}\{1\}$$

together with adequately behaving composition functions

$$\begin{aligned} M_0 \times M_0 &\longrightarrow M_0 & \qquad M_0 \times M_1 &\longrightarrow M_0 \\ M_1 \times M_0 &\longrightarrow M_0 & \qquad M_1 \times M_1 &\longrightarrow M_1 \end{aligned}$$

Therefore, $\mathbf{Mon}/U$ corresponds to an algebraic theory with two sorts.

The theory of the slice

For any T and any model $X \in \mathbf{Mod}(T)$, we define $X \backslash T$ as the pullback

$$\begin{array}{ccc} X\backslash T & \longrightarrow & \mathbf{Set}_* \\ \big\downarrow & \lrcorner & \big\downarrow \\ T & \underset{X}{\longrightarrow} & \mathbf{Set} \end{array}$$

We get an equivalence of categories

$$\mathbf{Mod}(X\backslash T) \quad \cong \quad \mathbf{Mod}(T)/X$$

Applying the free construction

We want to show that the functor

$$\mathbf{Sgp} \;\cong\; \mathbf{Mod}(T_{\mathbf{Sgp}}) \quad\longrightarrow\quad \mathbf{Mon}/U \;\cong\; \mathbf{Mod}(U\backslash T_{\mathbf{Mon}})$$

has a left adjoint.

For this, we show that this functor is forgetful, i.e. that it is the precomposition by some product-preserving functor

$$U\backslash T_{\mathbf{Mon}} \;\longrightarrow\; T_{\mathbf{Sgp}}$$

hence it has a left adjoint given by left Kan extension.

The internal model of prime ideals

For any semigroup S, the monoid homomorphism

$$p : \begin{array}{ccc} S \sqcup \{1\} & \longrightarrow & U := \{0,1\} \\ s & \longmapsto & 0 \\ 1 & \longmapsto & 1 \end{array}$$

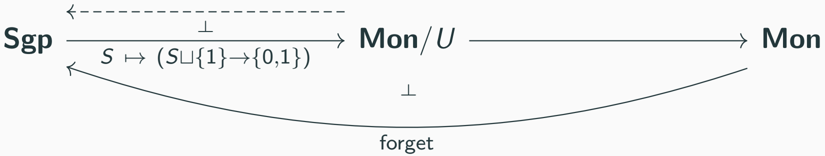
corresponds in the indexed way to the two sets

$$\begin{array}{cccc} (S \sqcup \{1\})_0 & := & S & \cong & \mathbf{Sgp}(\mathbb{N}^*, S) \\ (S \sqcup \{1\})_1 & := & \{1\} & \cong & \mathbf{Sgp}(\varnothing, S) \end{array}$$

We get in this way a model of $U \backslash T_{\mathbf{Mon}}$ internal to $T_{\mathbf{Sgp}}$.

Parametric right adjoints from universal algebra

The left adjoint **Sgp** $\to$ **Mon** is itself a parametric right adjoint:



In the same way, the two left adjoints

$$\mathrm{Cl}_1 \quad : \quad \mathbf{Mon} \longrightarrow \mathbf{Clone} \qquad \text{and} \qquad \mathrm{Cl}_0 \quad : \quad \mathbf{Set} \longrightarrow \mathbf{Clone}$$

are parametric right adjoints.

Profinite completions of sets and monoids

Moreover, the adjunctions

$$\mathbf{Mon} \underset{C_1 \,\leftarrow\!\shortmid\, C}{\overset{\mathsf{Cl}_1}{\underset{\perp}{\rightleftarrows}}} \mathbf{Clone} \qquad \text{and} \qquad \mathbf{Set} \underset{C_0 \,\leftarrow\!\shortmid\, C}{\overset{\mathsf{Cl}_0}{\underset{\perp}{\rightleftarrows}}} \mathbf{Clone}$$

crucially restrict to finite structures.

Theorem. The profinite completion of clones generalize the one of monoids:

$$\widehat{\mathsf{Cl}_1(M)} \quad \cong \quad \mathsf{Cl}_1(\widehat{M}) \qquad\qquad \text{for any monoid } M$$

Theorem. The profinite completion of clones generalize the ultrafilter monad:

$$\widehat{\mathsf{Cl}_0(X)} \quad \cong \quad \mathsf{Cl}_0(\beta X) \qquad\qquad \text{for any set } X$$

Conclusion

Much more to say:

- The algebraic viewpoint shows that clones and cartesian closed categories are closely related, cf. the work of Fiore, Mahmoud and Arkor.
- We have a third theorem relating the profinite completion of free clones to the profinite λ-calculus, a compactification of the syntax of cartesian closed categories.
- All details available in the last chapter of my PhD thesis, defended last week!

Future work: given a product-preserving functor $F : T \to T'$, when is the free construction functor $\mathrm{Lan}_F : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ a parametric right adjoint?

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Thank you for your attention!

Any questions?